高密度QCD理論や格子QCD計算の最前線

理研、量子ハドロン物理学研究室
土居　孝寛

自己紹介
- 専門：格子QCD
- 興味：QCD真空, カラーの閉じ込め, **符号問題**
- Heavy Ion Pub/Café : 初めまして

第35回Heavy Ion Cafe並びに第27回Heavy Ion Pub合同研究会「ポストQM2018 」、名古屋大学、2018 6/30

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有限密度格子QCDについて皆さんが知りたい事(予想)

Q. 有限密度格子QCDでQCD臨界点の兆候は既に見えているの？

A. 見えていません。

Q. 有限密度格子QCDって何が計算できているの？

A. 基本的には高温・低密度領域でしか計算できていません。
このトークで基本的な事をレビューします。

Q. 最近complex Langevin methodとか
Lefschetz thimbleとかよく聞くけど、どうなの？

A. ここ数年で理論的進展がありましたが、
まだ結果が出始めていた所です。

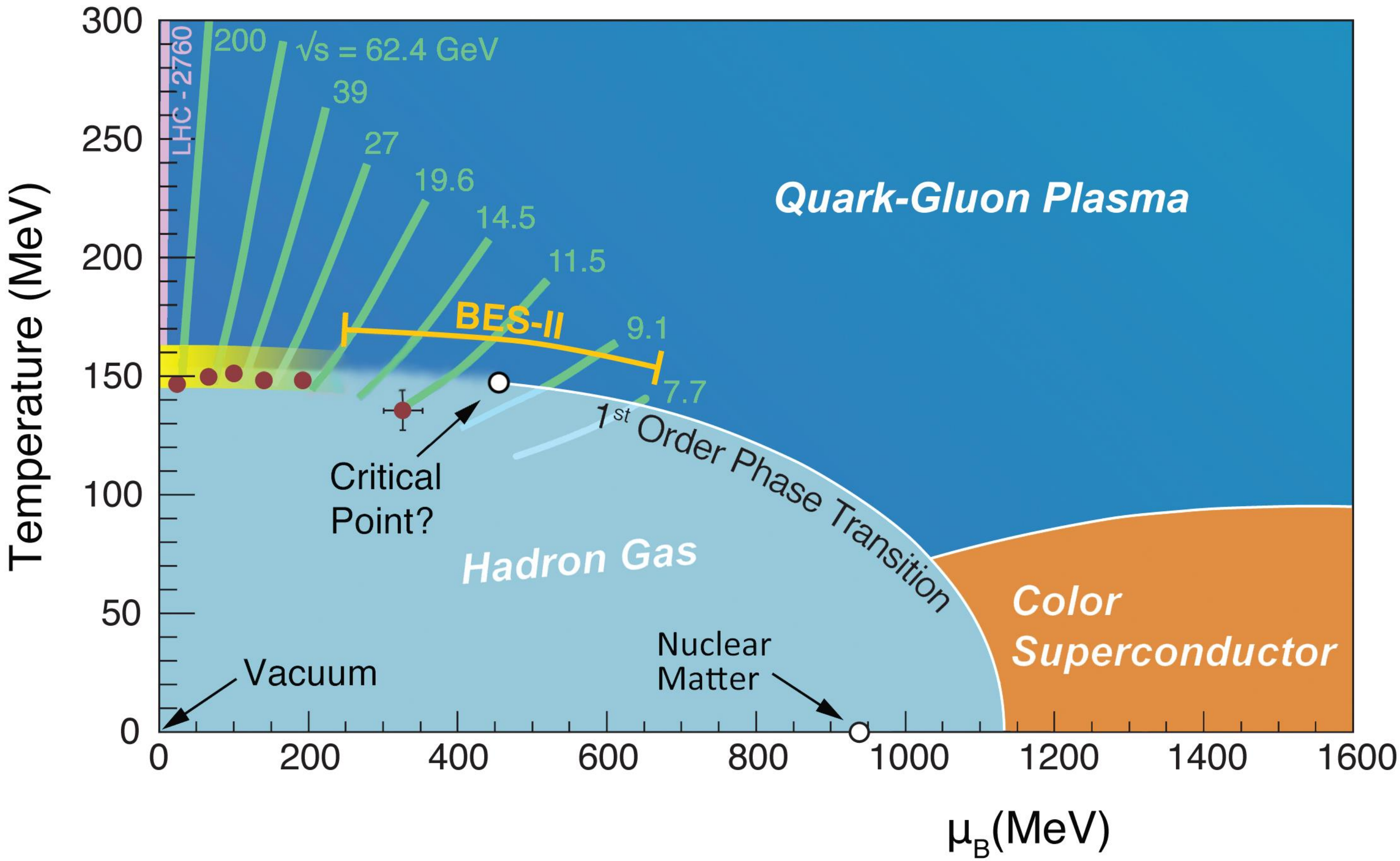
Q. QCDの符号問題っていつ解けるの？

A. わかりません・・・

Contents of this talk

Introduction

Busza, Rajagopal, Schee, arXiv: 1802.04801



Introduction

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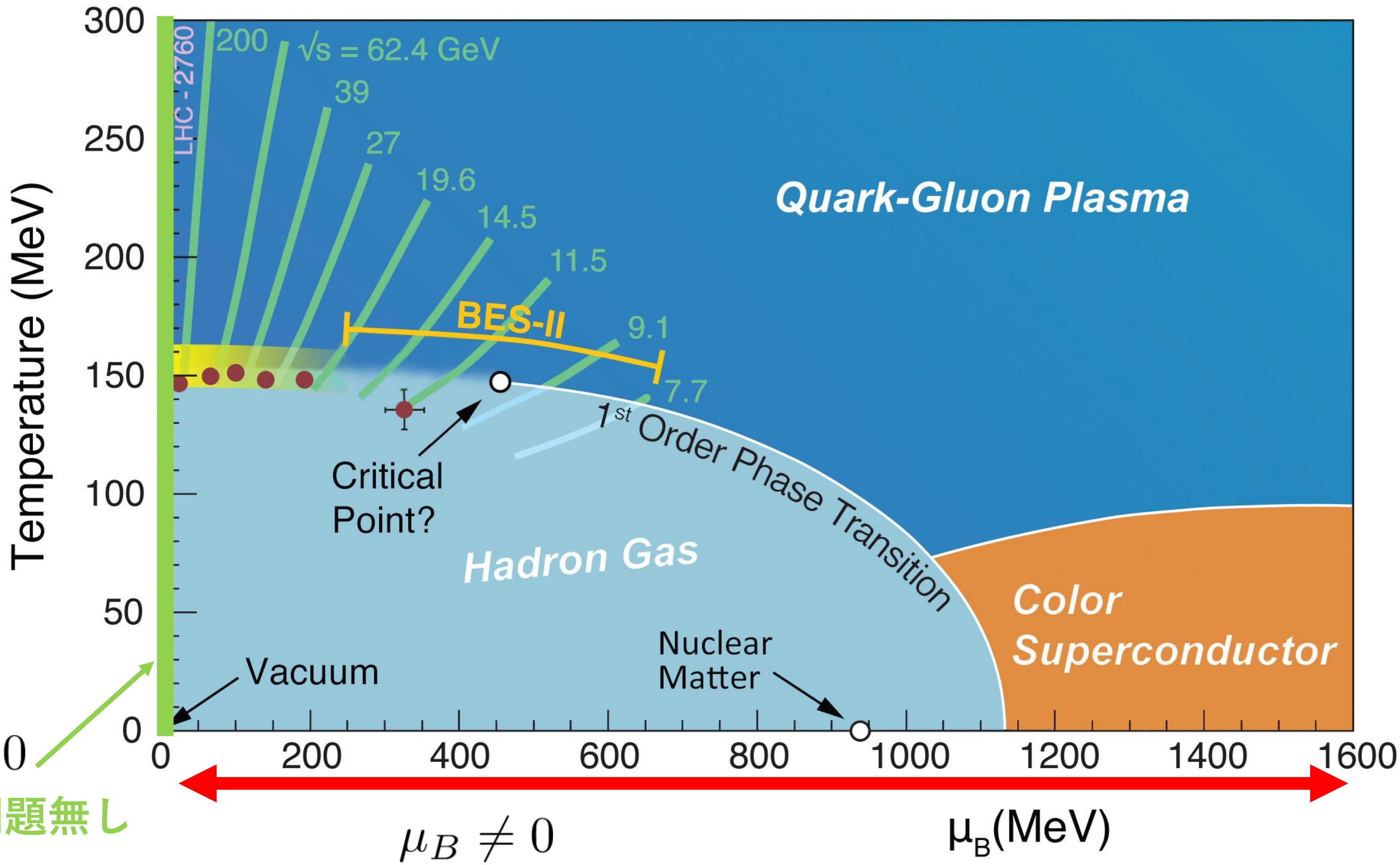


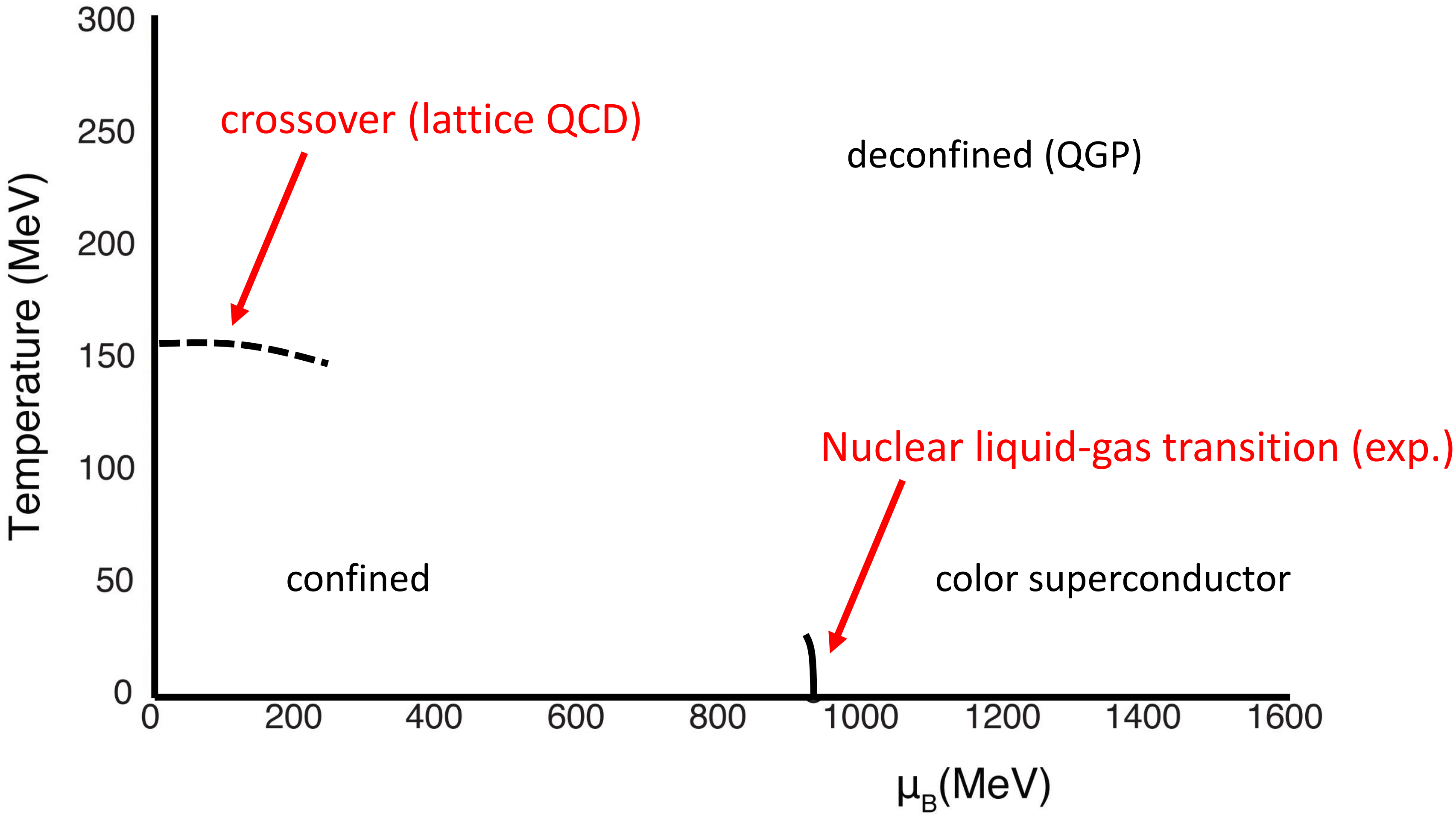
$\mu_B = 0$

符号問題無し

$\mu_B \neq 0$

符号問題あり

Introduction



Monte Carlo法

分配関数

$$Z = \int dx P(x)$$

x: 積分変数
(e.g. QCD: gluon and fermion filed)

P(x): 分布関数 $P(x) = \mathrm{e}^{-S(x)}$

物理量の平均期待値

$$\langle \mathcal{O} \rangle = \frac{\int dx \mathcal{O}(x) P(x)}{\int dx P(x)}$$

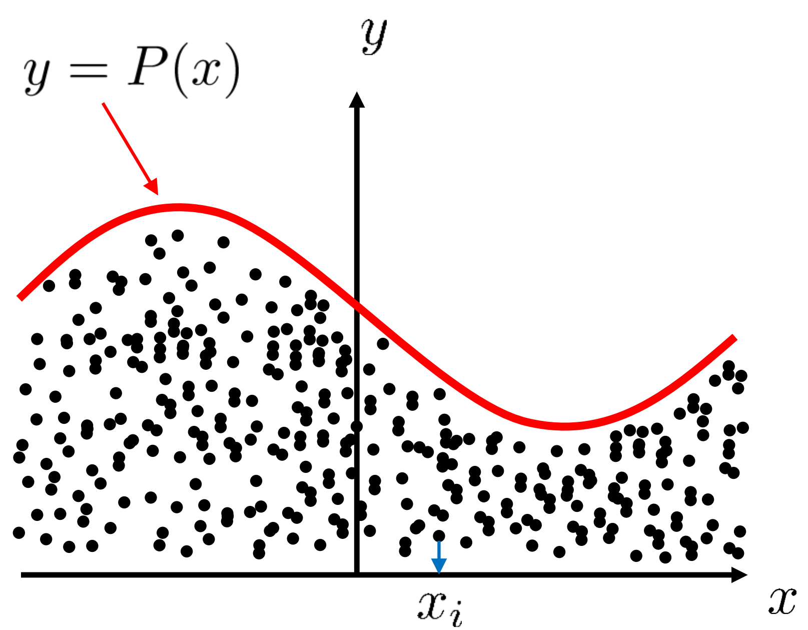
Monte Carlo simulation(Importance sampling)

$$\langle \mathcal{O} \rangle \simeq \sum_{i=1}^{N} \mathcal{O}(x_i)$$

統計誤差：$\sim 1/\sqrt{N}$

現在知られている一般的な計算手法の中で最も効率が良く、非常に強力

例：１次元積分

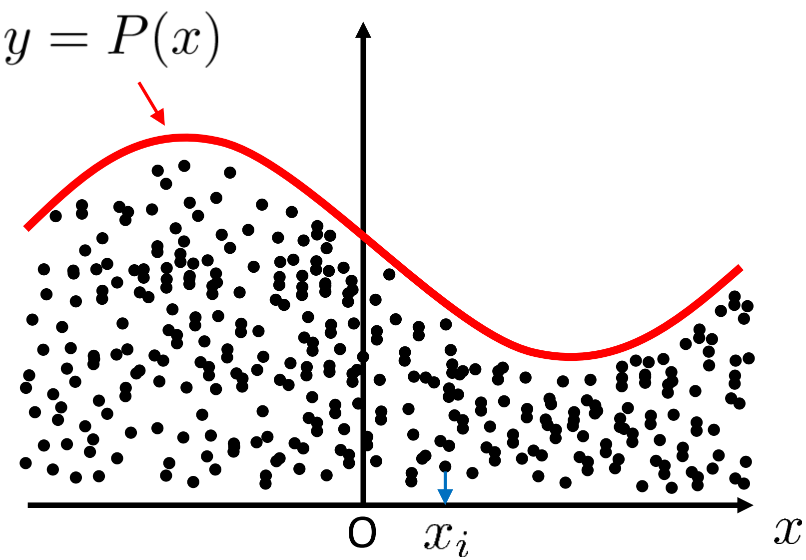


Monte Carlo法

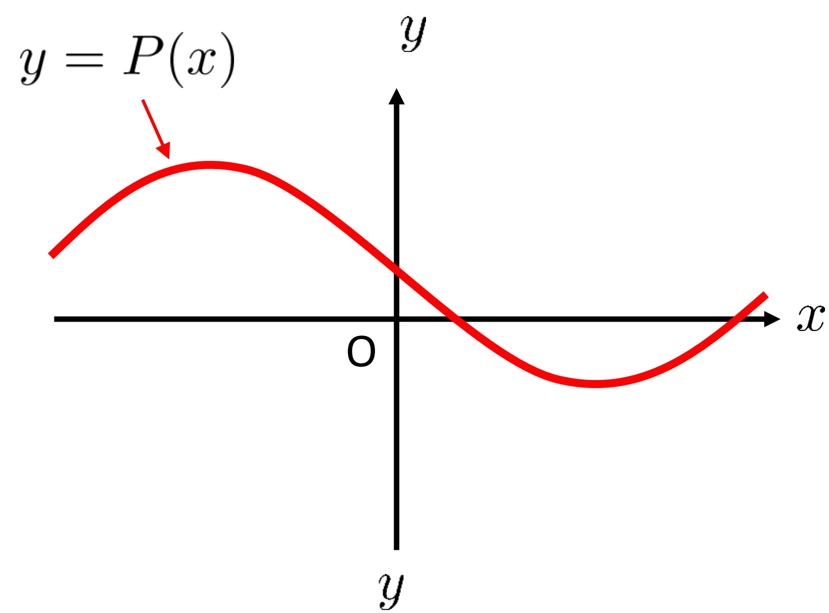
$$\langle \mathcal{O} \rangle = \frac{\int dx \mathcal{O}(x) P(x)}{\int dx P(x)} \simeq \sum_{i=1}^{N} \mathcal{O}(x_i)$$

Monte Carlo simulation(Importance sampling)が正当化される条件: $P(x) > 0$

例： 1 次元積分



$P(x) > 0$ の場合



$P(x)$ の符号が決まっていない場合

振動関数の積分＝“振動積分”

Sign problem in (lattice) QCD

Euclidean QCD partition function

$$Z = \int \mathcal{D}A [\det(\gamma_\mu D_\mu + m_{\rm q} + \mu\gamma_4)]^{N_f} e^{-S_{\rm YM}}$$

$$S_{\rm YM} = \int d^4x \frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu} \in \mathbb{R}$$: pure gluonic (Yang-Mills) action ⇒ $e^{-S_{\rm YM}} > 0$

· $\mu = 0$ (for example, finite-temperature lattice QCD)

$$\det(\gamma_\mu D_\mu + m_{\rm q}) \in \mathbb{R}$$

⇒ $[\det(\gamma_\mu D_\mu + m_{\rm q})]^{N_f} > 0$ with even N_f

On the lattice, importance sampling (Monte Carlo method) can be done using the probability distribution $[\det(\gamma_\mu D_\mu + m_{\rm q} + \mu\gamma_4)]^{N_f} e^{-S_{\rm YM}} > 0$.

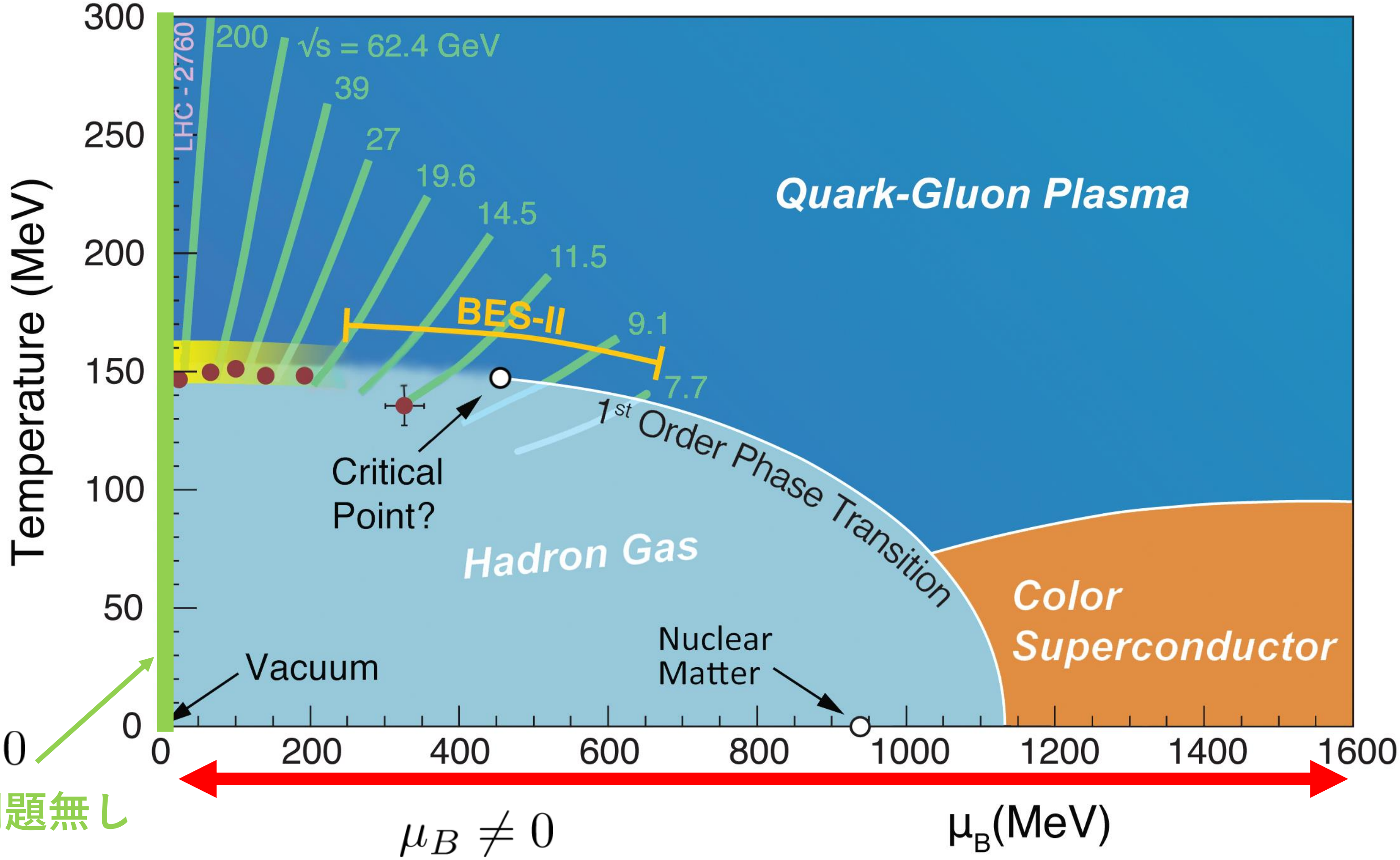
· $\mu \neq 0 \ \ (\mu \in \mathbb{R})$

$$\det(\gamma_\mu D_\mu + m_{\rm q} + \mu\gamma_4) \in \mathbb{C}$$

$[\det(\gamma_\mu D_\mu + m_{\rm q} + \mu\gamma_4)]^{N_f} e^{-S_{\rm YM}}$ cannot be regarded as a probability distribution. Thus the Monte Carlo method is not applicable in general.

“符号問題” (=負符号問題, 複素位相問題)

Busza, Rajagopal, Schee, arXiv: 1802.04801



$\mu_B = 0$

符号問題無し

$\mu_B \neq 0$

符号問題あり

「有限密度領域で格子QCDの研究をする事」=「符号問題をどうにかする事」

Contents of this talk

１． introduction: 有限密度QCDと符号問題

２． 符号問題の回避・解決方法の紹介(QM2018の結果含む)

- **再重み付け法(reweighting method)**
- **Taylor展開法**
- **pure imaginary chemical potential法**
- **complexifying integration path**
- **complex Langevin method**

３． まとめ

Many proposed methods to solve(avoid) the sign problem

Proposed methods	Based on
Taylor expansion method	Monte Carlo method
Canonical approach	
Reweighting method	
with imaginary chemical potential	
$\mathrm{SU}(2)_\mathrm{c}$ **QCD**	
Lefschetz thimble decomposition	Cauchy's integral formula
Holomorphic gradient flow method	
path optimization method	
Complex Langevin method	Stochastic quantization

and so on

Many proposed methods to solve(avoid) the sign problem

	Proposed methods	Based on
比較的古くから研究されている	**Taylor expansion method**	Monte Carlo method
	Canonical approach	
	Reweighting method	
	with imaginary chemical potential	
	$\mathrm{SU}(2)_c$ **QCD**	
ここ数年で理論的発展	**Lefschetz thimble decomposition**	Cauchy's integral formula
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and so on

再重み付け法(reweighting method)

特徴

長所　・原理的に正しい符号問題の解法
　　　・符号問題の難しさがよくわかる方法（個人的意見）

短所　・符号問題が厳しくなると実用的でなくなる

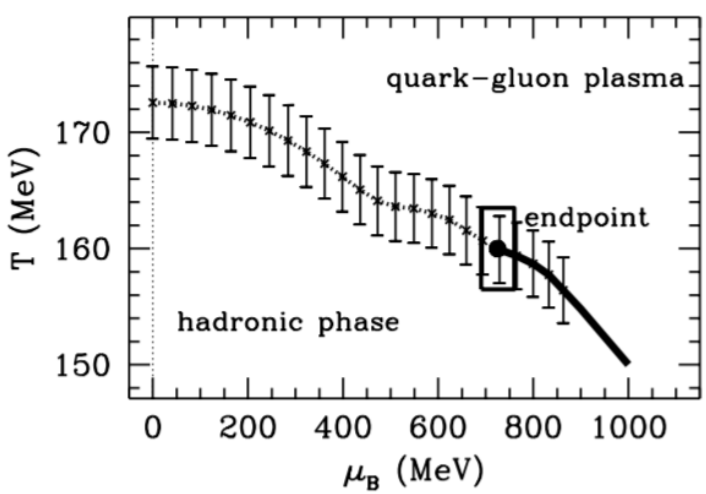


Figure 2: The phase diagram in physical units. Direct results are with errorbars. Dotted line illustrates the crossover, solid line the first order phase transition. The small box shows the uncertainties of the endpoint.

$\mu_{\rm B}$ が大きくなるにつれて
計算の精度が悪くなり信用できなくなるので、
左図のcritical endpointの位置が
正しいかどうかは非自明

Z. Fodor, S.D. Katz, JHEP 0203 (2002) 014.
C.R. Allton, S. Ejiri, et al., Phys.Rev. D66 (2002) 074507.
Z. Fodor, S.D. Katz, Phys.Lett. B534 (2002) 87-92.

再重み付け法(reweighting method)

分配関数 $Z=\int dx P(x)=\int dx \frac{P(x)}{|P(x)|}|P(x)|=\int dx \mathrm{e}^{i\theta}|P(x)|$

物理量の平均期待値

$$\langle \mathcal{O} \rangle_P = \frac{\int dx \mathcal{O}(x) P(x)}{\int dx P(x)} = \frac{\int dx \mathcal{O}(x) \mathrm{e}^{i\theta}|P(x)|}{\int dx \mathrm{e}^{i\theta}|P(x)|} = \frac{\frac{\int dx \mathcal{O}(x)\mathrm{e}^{i\theta}|P(x)|}{\int dx |P(x)|}}{\frac{\int dx \mathrm{e}^{i\theta}|P(x)|}{\int dx |P(x)|}} = \frac{\langle \mathcal{O}\mathrm{e}^{i\theta}\rangle_{|P|}}{\langle \mathrm{e}^{i\theta}\rangle_{|P|}}$$

$\langle \mathcal{O}\mathrm{e}^{i\theta}\rangle_{|P|}$, $\langle \mathrm{e}^{i\theta}\rangle_{|P|}$

はMonte Carlo法で計算可能

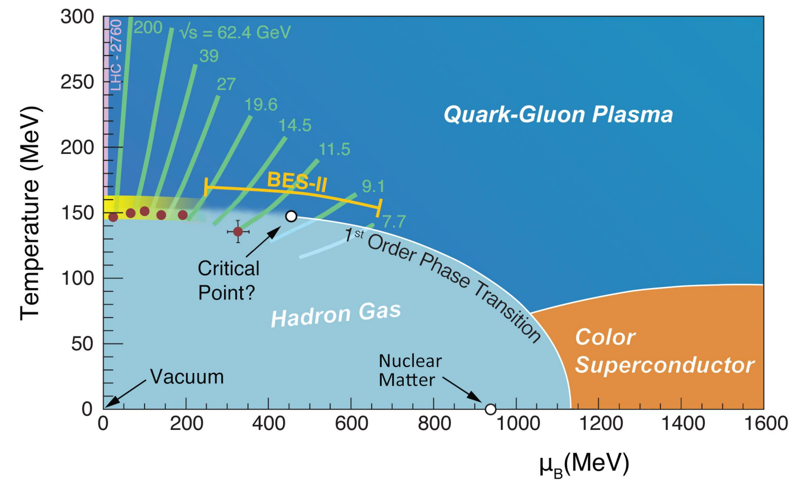
$\because \ |P(x)|>0$

↓

よって、再重み付け法は符号問題
(=Monte Carlo法が適用できないという問題)
を回避できている！

ここで紹介している方法はreweigtingの一種で、
正確には“Phase quenching reweigting method”という。
すごい点：このスライド上の「=」は全てexactな
等号であって、一切の近似が入っていない

再重み付け法の欠点



$$\langle \mathcal{O} \rangle_P = \frac{\langle \mathcal{O} e^{i\theta} \rangle_{|P|}}{\langle e^{i\theta} \rangle_{|P|}}$$

平均位相因子(Average phase factor)の性質

$$\langle e^{i\theta} \rangle_{|P|} = \frac{\int dx e^{i\theta} |P(x)|}{\int dx |P(x)|} = \frac{Z_P}{Z_{|P|}} = \exp\left(-\frac{V}{T}\Delta F\right)$$

ΔF : P(x)の理論と|P(x)|の理論の自由エネルギーの差
μ大⇔ΔF大

再重み付け法の問題点

× 体積や化学ポテンシャルを大きくしたり(V, μ→大)、低温(T→小)を考えると、平均位相因子 $\langle e^{i\theta} \rangle_{|P|}$ は指数関数的に0に近づく

⇒計算時間が指数関数的に長くなる
⇔**物理的に妥当な状況や、興味のある系の計算は非常に難しい**

Many proposed methods to solve(avoid) the sign problem

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Canonical approach	
Reweighting method	
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path optimization method	
Complex Langevin method	Stochastic quantization

どちらも低密度(small μ)
でしか正当化されないが、
確実な方法

正しくはsmall μ/T

and so on

Taylor展開法

基本的なidea

物理量（例：圧力 $P(T,\mu)$）を μ/T で展開して、係数を $\mu=0$ で計算する

$$P(T,\mu)-P(T,0)=\sum_k c_{2k}(T)\left(\frac{\mu}{T}\right)^{2k}$$

計算すべき量

$c_{2k}=\langle\mathrm{Tr}($ $M,\ \frac{\partial M}{\partial\mu}$ の4k次の多項式 $)\rangle_{\mu=0}$ ← Monte Carlo法で計算可能

M : Dirac operator $(S_q=\bar{q}Mq)$

特徴

長所　・絶対に正しい in 収束半径

短所　・収束半径を超えると適用不可能($\mu/T\geq 1$ くらい？)
・高次の係数の計算コストが高すぎる
・⇒収束半径の見積もり(高次係数が必要)もかなり難しい

c_8：out pf reach
c_4: 2002
c_6: 2005

From talk by Philip De Forcrand @NFQCD2018

Taylor expansion: nitty-gritty

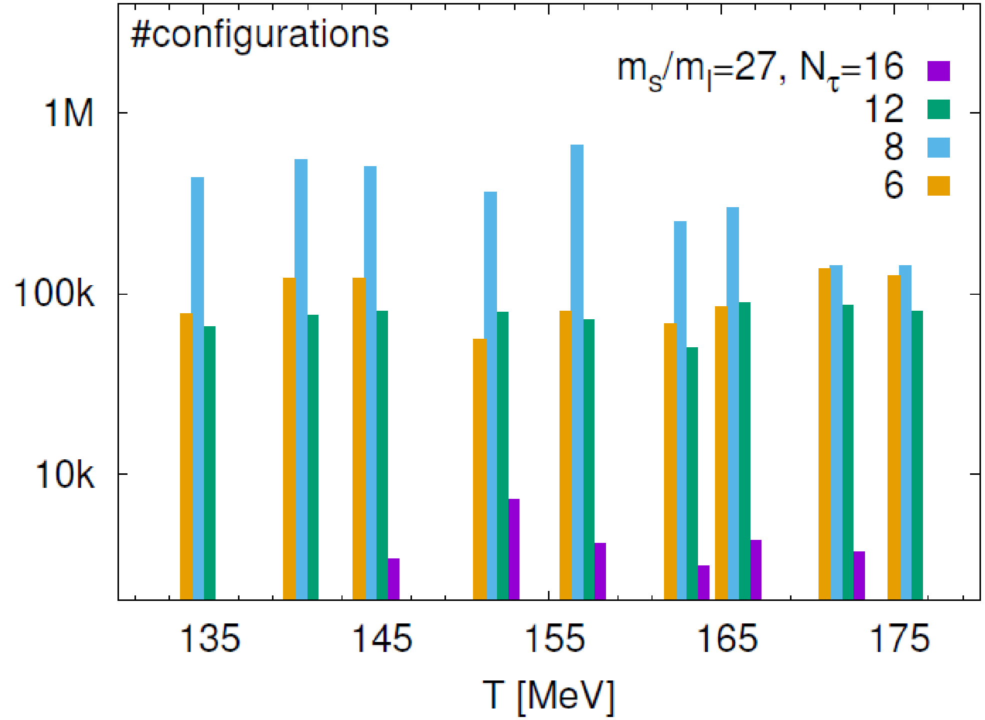
$$
\begin{aligned}
\frac{\partial^6 \ln \det M}{\partial \mu^6} = & \operatorname{tr}\left(M^{-1}\frac{\partial^6 M}{\partial \mu^6}\right) - 6\operatorname{tr}\left(M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial^5 M}{\partial \mu^5}\right) \\
& -15\operatorname{tr}\left(M^{-1}\frac{\partial^2 M}{\partial \mu^2}M^{-1}\frac{\partial^4 M}{\partial \mu^4}\right) - 10\operatorname{tr}\left(M^{-1}\frac{\partial^3 M}{\partial \mu^3}M^{-1}\frac{\partial^3 M}{\partial \mu^3}\right) \\
& +30\operatorname{tr}\left(M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial^4 M}{\partial \mu^4}\right) + 60\operatorname{tr}\left(M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial^2 M}{\partial \mu^2}M^{-1}\frac{\partial^3 M}{\partial \mu^3}\right) \\
& +60\operatorname{tr}\left(M^{-1}\frac{\partial^2 M}{\partial \mu^2}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial^3 M}{\partial \mu^3}\right) + 30\operatorname{tr}\left(M^{-1}\frac{\partial^2 M}{\partial \mu^2}M^{-1}\frac{\partial^2 M}{\partial \mu^2}M^{-1}\frac{\partial^2 M}{\partial \mu^2}\right) \\
& -120\operatorname{tr}\left(M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial^3 M}{\partial \mu^3}\right) \\
& -180\operatorname{tr}\left(M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial^2 M}{\partial \mu^2}M^{-1}\frac{\partial^2 M}{\partial \mu^2}\right) \\
& -90\operatorname{tr}\left(M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial^2 M}{\partial \mu^2}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial^2 M}{\partial \mu^2}\right) \\
& +360\operatorname{tr}\left(M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial^2 M}{\partial \mu^2}\right) \\
& -120\operatorname{tr}\left(M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial M}{\partial \mu}\right).
\end{aligned}
$$

Now estimate all Traces by sandwiching between noise vectors... GPUs

From talk by Patrick Steinbrecher (HotQCD Collaboration) ＠QM2018

Quantum Chromodynamics
from first principles

- Lattice QCD
  - HISQ action
  - $N_\sigma = 4N_\tau$
  - sim. at $\mu = 0$
- physical quarks
  - 2 light quarks
  - 1 strange quark
  - $m_s/m_l = 27$
- $m_\pi \simeq 138\ \mathrm{MeV}$



→ everything continuum extrapolated

要約：かなり現実に近いパラメータセットで信頼できる計算をしましたよ

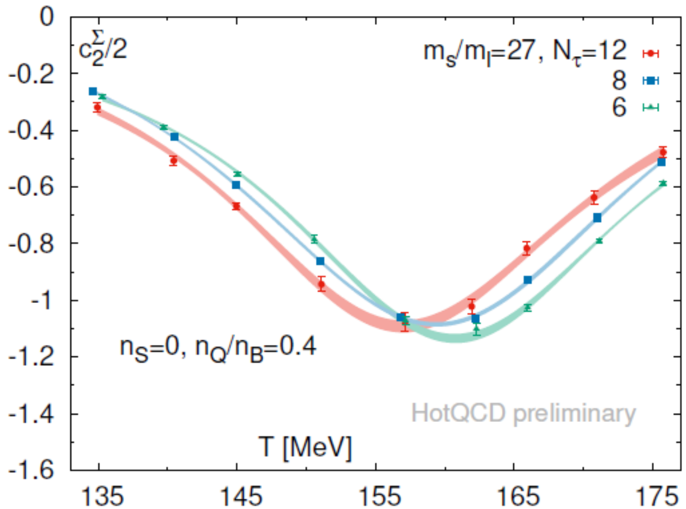
From talk by Patrick Steinbrecher (HotQCD Collaboration) @QM2018

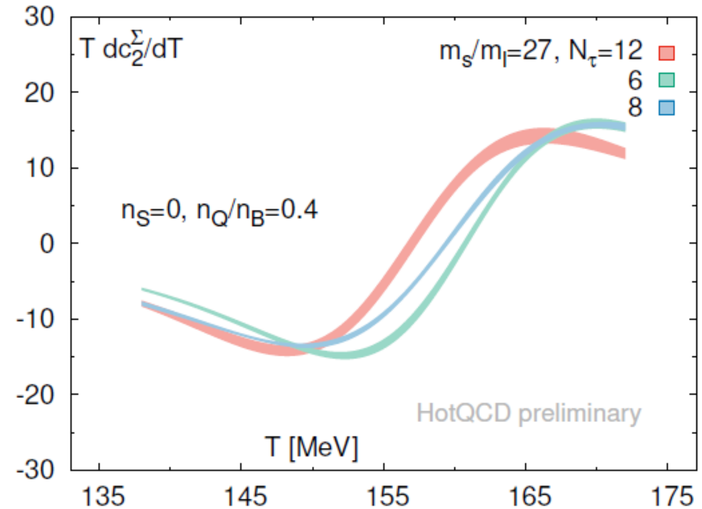
- subtracted condensate

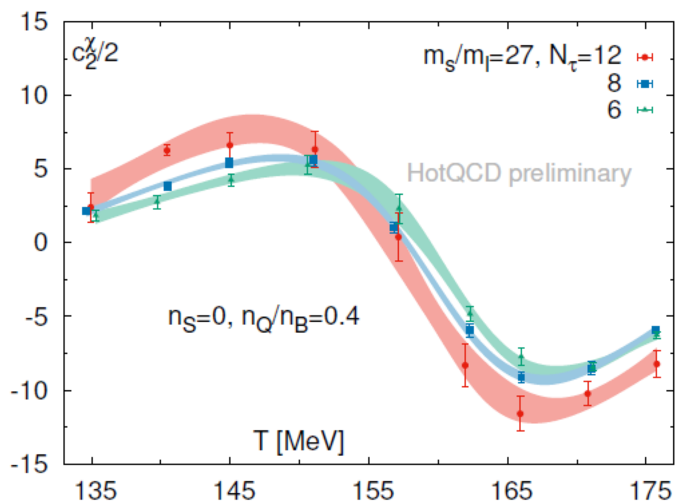
$$\frac{\Sigma_{\rm sub}}{f_K^4} = \sum_{n=0}^{\infty} \frac{c_n^{\Sigma}}{n!} \hat{\mu}_B^n \qquad \text{with} \qquad c_n^{\Sigma} = \left. \frac{\partial \Sigma_{\rm sub}/f_K^4}{\partial \hat{\mu}_B^n} \right|_{\mu=0}$$

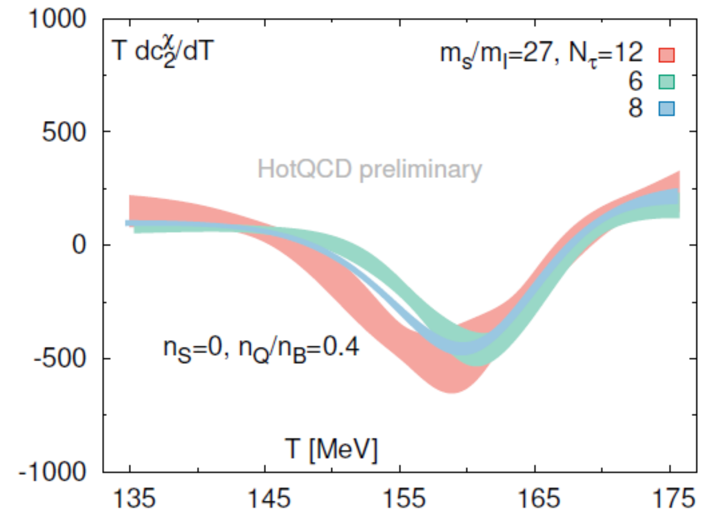
- disconnected susceptibility

$$\frac{\chi_{\rm disc}}{f_K^4} = \sum_{n=0}^{\infty} \frac{c_n^{\chi}}{n!} \hat{\mu}_B^n \qquad \text{with} \qquad c_n^{\chi} = \left. \frac{\partial \chi_{\rm disc}/f_K^4}{\partial \hat{\mu}_B^n} \right|_{\mu=0}$$









The curvature of the crossover line

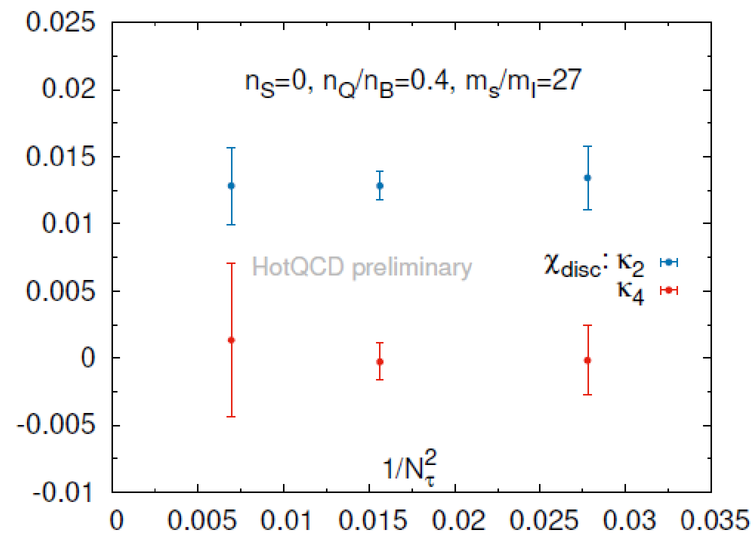
$$\frac{T_c(\mu_B)}{T_0} = 1 - \kappa_2 \left(\frac{\mu_B}{T_0}\right)^2 - \kappa_4 \left(\frac{\mu_B}{T_0}\right)^4 + \mathcal{O}(\mu_B^6)$$

- Taylor expansion in μ and T of:

$$\frac{d}{dT}\frac{\chi_{\rm disc}(T,\mu_B)}{f_K^4} = (\ldots)\mu_B^2 + (\ldots)\mu_B^4 + \ldots = \mathbf{0}$$

- has to be zero order by order

$$\kappa_2 = \frac{1}{2T_0^2}\frac{T_0 \left.\frac{\partial c_2^\chi}{\partial T}\right|_{(T_0,0)} - 2\, c_2^\chi\big|_{(T_0,0)}}{\left.\frac{\partial^2 c_0^\chi}{\partial T^2}\right|_{(T_0,0)}}$$

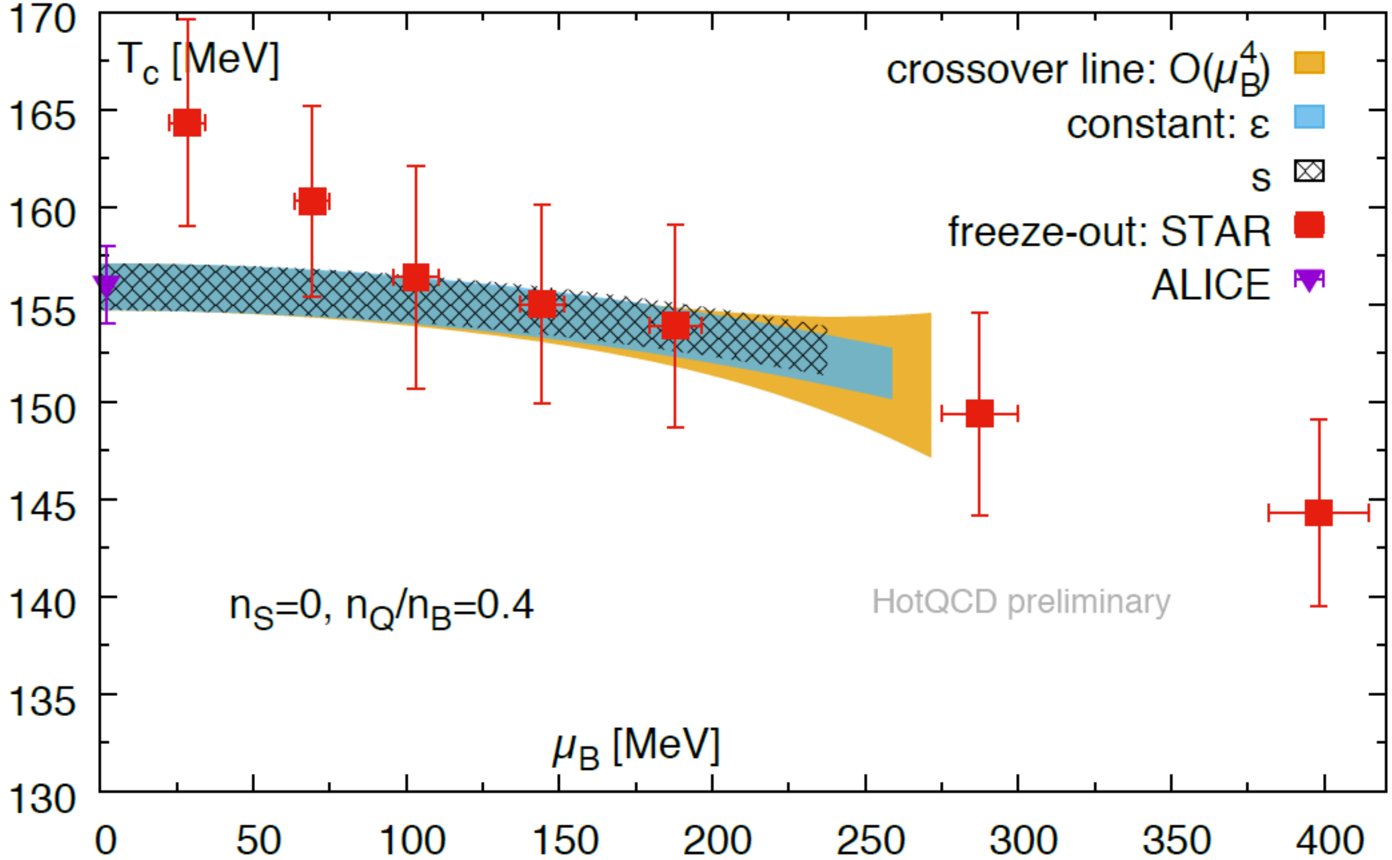


From talk by Patrick Steinbrecher (HotQCD Collaboration) @QM2018

The QCD crossover line

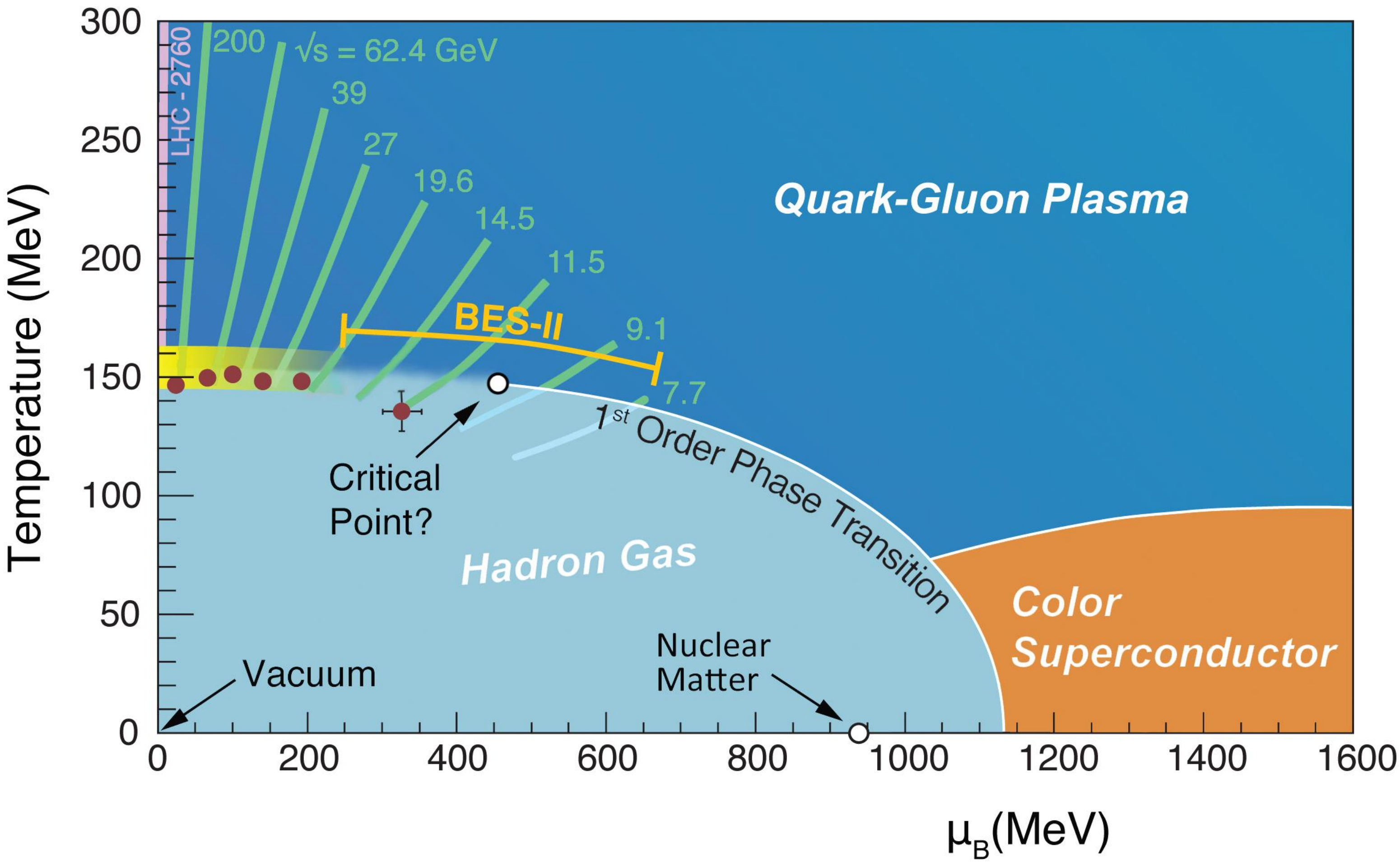
STAR: arxiv:1701.07065

ALICE: arxiv:1408.6403



- Crossover along const. entropy density and energy density
- Chemical freeze-out might be close to crossover

QCD critical point search by Taylor expansion method



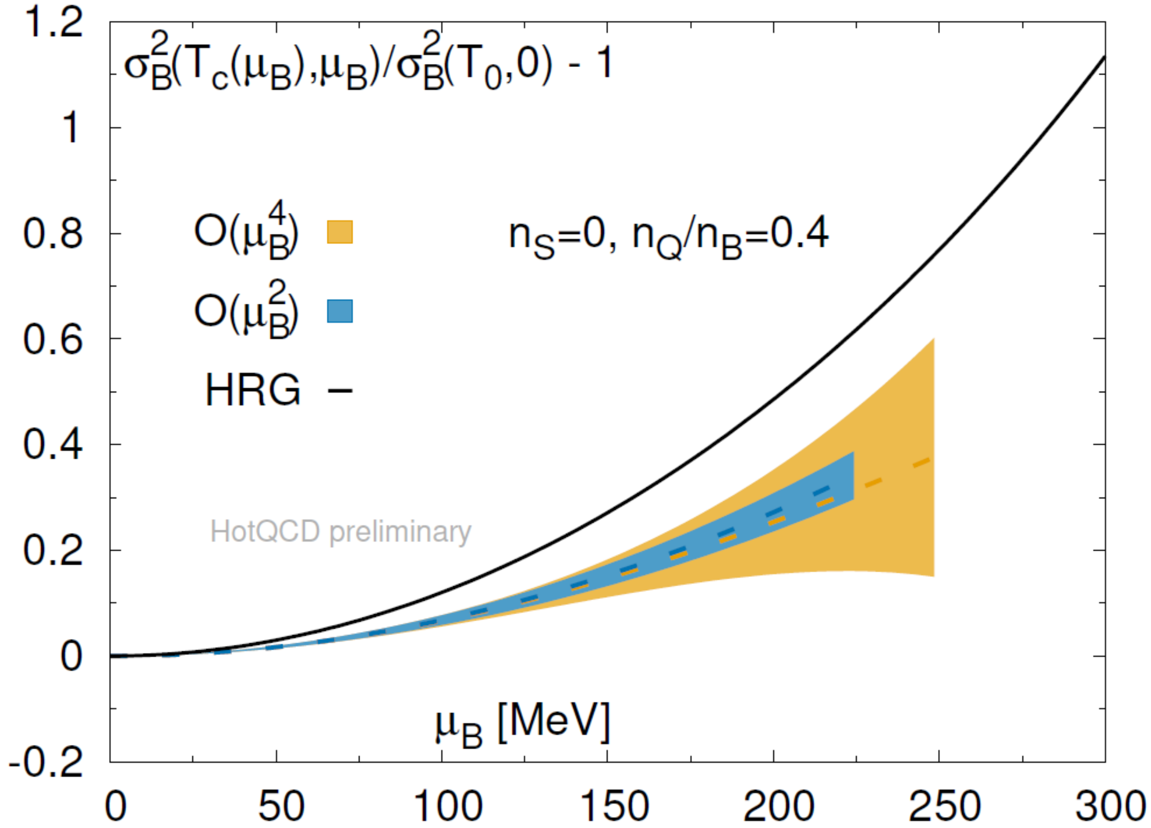
もし、crossover line上でバリオン数ゆらぎの急激な変化があれば

⇩

QCD critical pointに近づいている証拠

From talk by Patrick Steinbrecher (HotQCD Collaboration) @QM2018

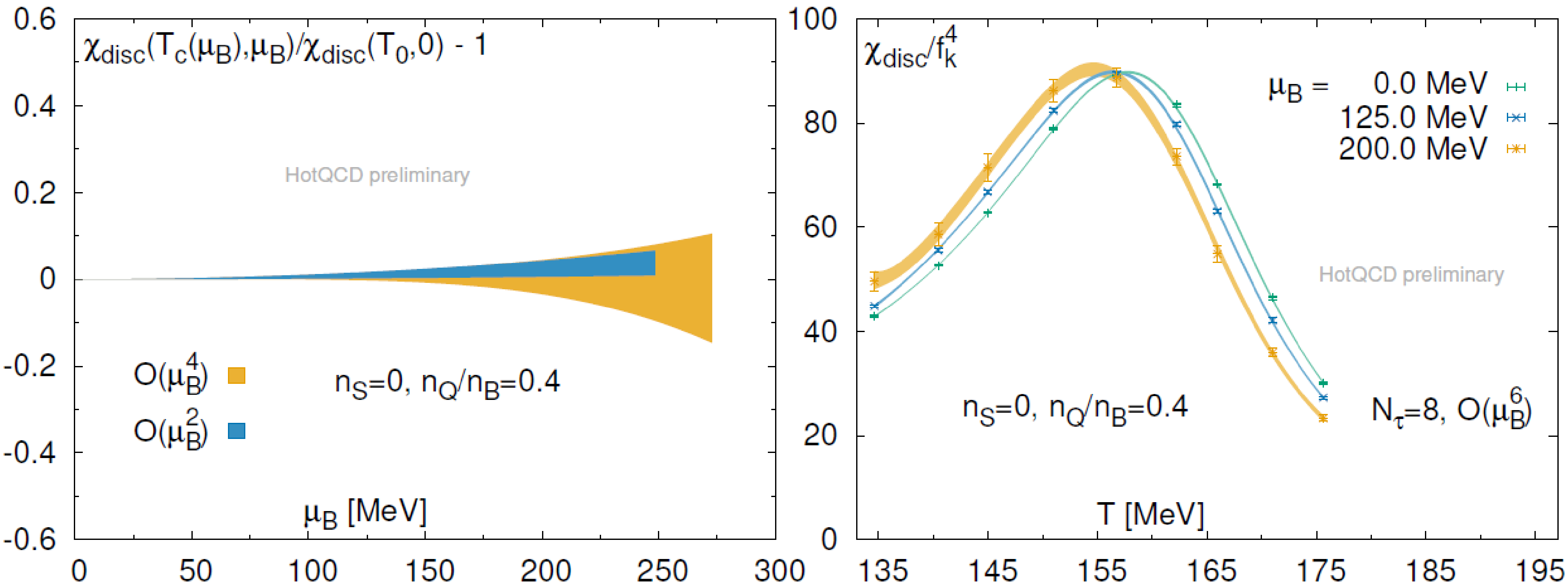
Baryon-number fluctuations ⇝ along $T_c(\mu_B)$



$$\frac{\sigma_B^2(T_c(\mu_B),\mu_B) - \sigma_B^2(T_0,0)}{\sigma_B^2(T_0,0)} = \lambda_2 \left(\frac{\mu_B}{T_0}\right)^2 + \lambda_4 \left(\frac{\mu_B}{T_0}\right)^4 + \cdots$$

σ_{B} : Baryon number fluctuation

Susceptibility fluctuations $\rightsquigarrow$ along $T_c(\mu_B)$



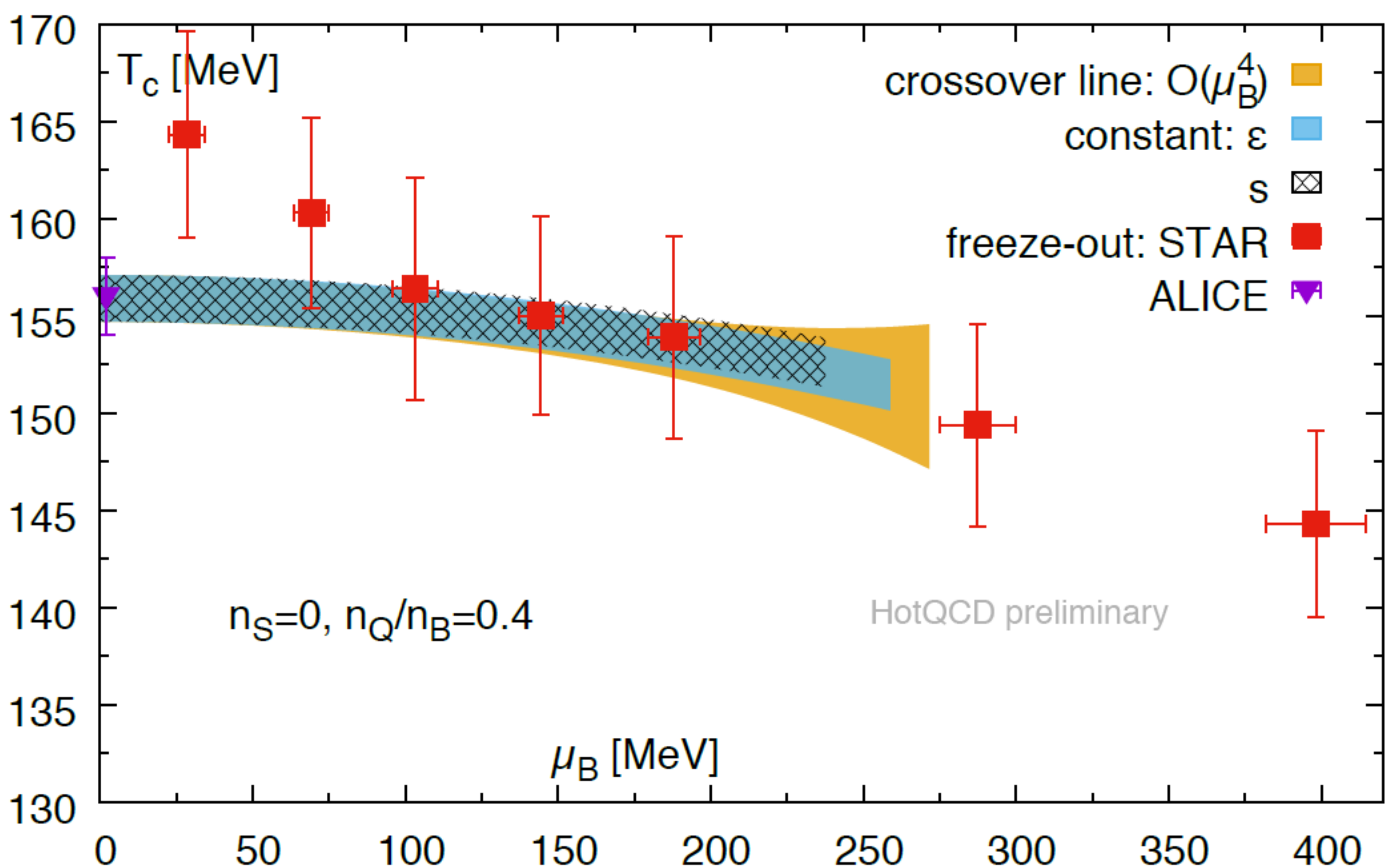
σ_B^2 and $\chi_{\rm disc}$ show no indication that crossover gets stronger

From talk by Patrick Steinbrecher (HotQCD Collaboration) @QM2018

The QCD crossover line

STAR: arxiv:1701.07065

ALICE: arxiv:1408.6403

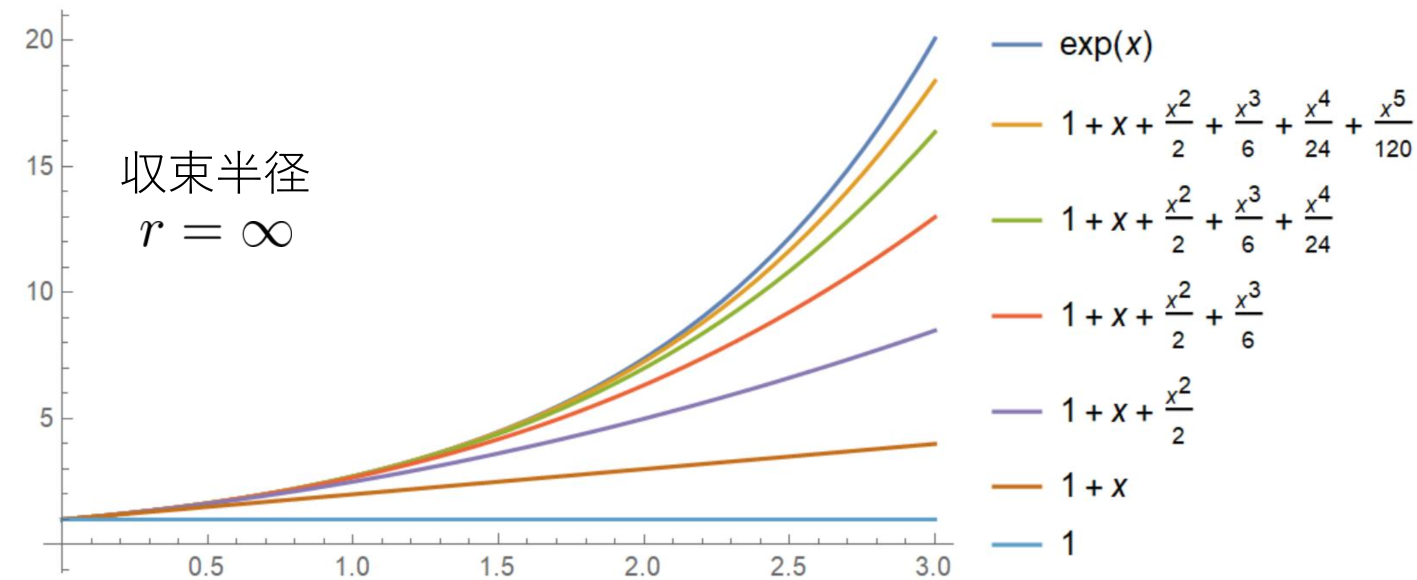


$\mu_{\rm B}/T \geq 1$ でもplotしているが、よいのだろうか・・・

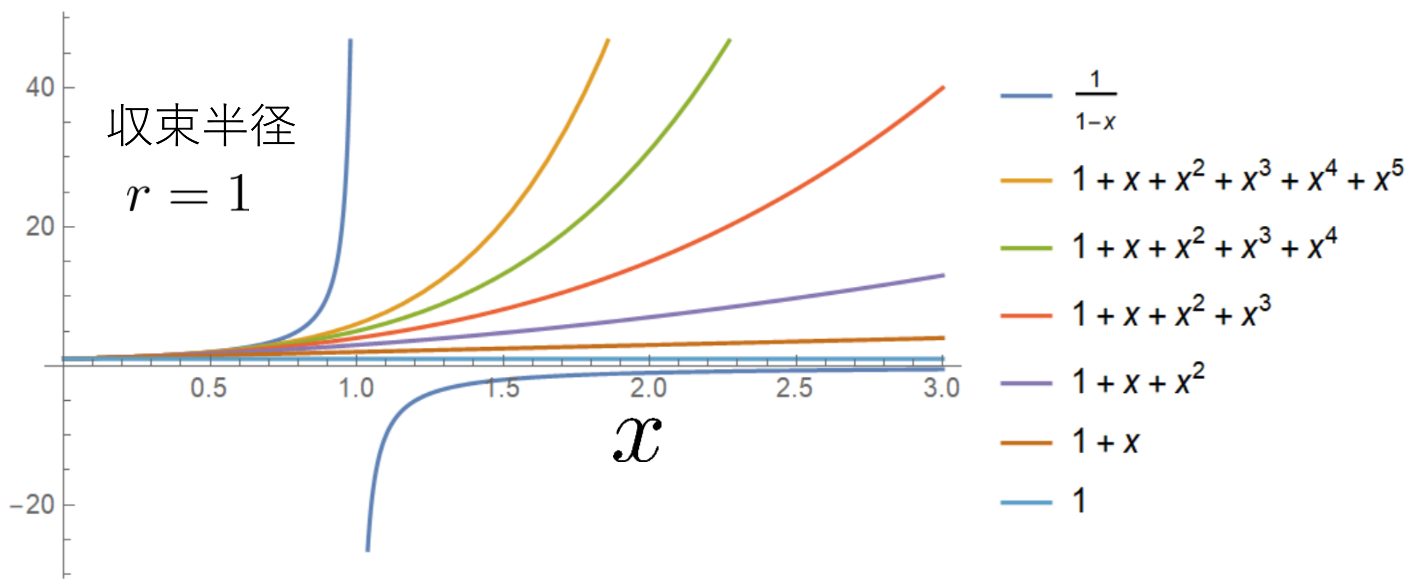
$$\frac{T_c(\mu_B)}{T_0} = 1 - \kappa_2 \left(\frac{\mu_B}{T_0}\right)^2 - \kappa_4 \left(\frac{\mu_B}{T_0}\right)^4 + \mathcal{O}(\mu_B^6)$$

by mathematica

参考：e^{x} のTaylor expansion around $x = 0$



参考：$\frac{1}{1-x}$ のTaylor expansion around $x = 0$



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$\mathrm{SU}(2)_c$ **QCD**	
Lefschetz thimble decomposition	
Holomorphic gradient flow method	Cauchy's integral formula
path optimization method	
Complex Langevin method	Stochastic quantization

どちらも低密度(small μ)でしか正当化されないが、確実な方法

正しくはsmall μ/T

and so on

虚数化学ポテンシャル(imaginary chemical potential)

基本的なidea

化学ポテンシャル $\mu \in \mathbb{R}$ を無理やり純虚数 $\mu = i\mu_I \ (\mu_I \in \mathbb{R})$ にすると、
符号問題が生じない事を利用する：

$$Z_{\rm QCD}(\mu \to i\mu_I) = \int \mathcal{D}A[\det(\gamma_\mu D_\mu + m_{\rm q} + i\mu_I\gamma_4)]^{N_f} e^{-S_{\rm YM}}$$

$$\det(\gamma_\mu D_\mu + m_{\rm q} + i\mu_I\gamma_4) \in \mathbb{R}$$

戦略：虚数化学ポテンシャルで展開して解析接続する

$$\mathcal{O}(T, i\mu_I) - \mathcal{O}(T, 0) = \sum_k c_{2k}(T)\left(\frac{i\mu_I}{T}\right)^{2k}$$

$i\mu_I \to \mu \in \mathbb{R}$ ↓ 解析接続：同じ係数を使う

$$\mathcal{O}(T, \mu) - \mathcal{O}(T, 0) = \sum_k c_{2k}(T)\left(\frac{\mu}{T}\right)^{2k}$$

$\mathcal{O}(T, i\mu_I), \mathcal{O}(T, 0)$
はMonte Carlo法で計算可能

↓

fittingして係数 $c_{2k}(T)$
を求める

虚数化学ポテンシャル(imaginary chemical potential)

戦略：虚数化学ポテンシャルで展開して解析接続する

$$\mathcal{O}(T, i\mu_I) - \mathcal{O}(T, 0) = \sum_k c_{2k}(T) \left(\frac{i\mu_I}{T}\right)^{2k}$$

解析接続：同じ係数を使う

$$\mathcal{O}(T, \mu) - \mathcal{O}(T, 0) = \sum_k c_{2k}(T) \left(\frac{\mu}{T}\right)^{2k}$$

$\mathcal{O}(T, i\mu_I), \mathcal{O}(T, 0)$

はMonte Carlo法で計算可能

fittingして係数 $c_{2k}(T)$

を求める

特徴

長所
- Taylor展開法に比べると計算コストが低い
- $\mathcal{O}(T, i\mu_I)$ 自体はμ/Tの高次の効果の情報を持っている（はず）
- Taylor展開法とcross checkができる

短所
- 収束半径を超えると適用不可能 or 系統誤差がある
- 解析接続できる保証はない（でもたぶん大丈夫と思っている）

From talk by Attila Pasztor (Wuppertal-Budapest) ＠QM2018、arXiv:1805.04445

χ_{2k}^{B} ：バリオン数の2k次ゆらぎ

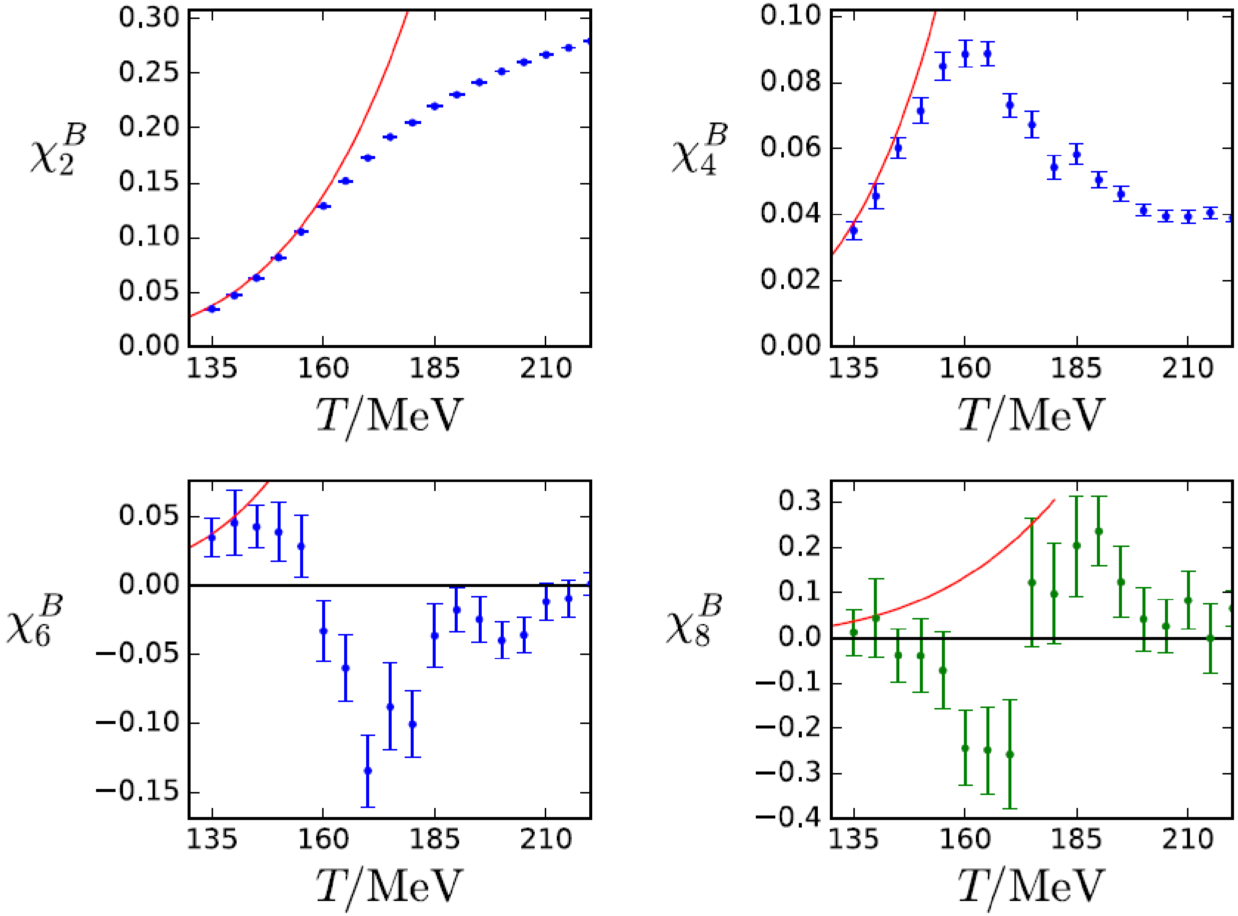
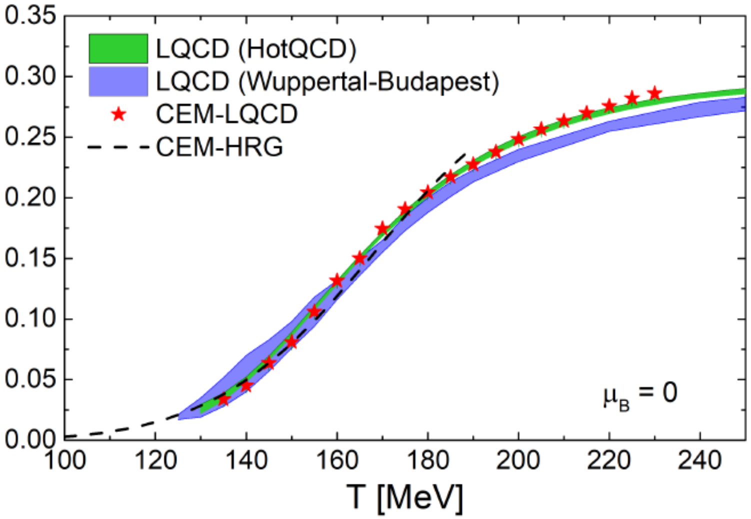


Figure 2. Results for χ_2^B, χ_4^B, χ_6^B and an estimate for χ_8^B as functions of the temperature, obtained from the single-temperature analysis. We plot χ_8^B in green to point out that its determination is guided by a prior, which is linked to the χ_4^B observable by Eq. (3.4). The red curve in each panel corresponds to the Hadron Resonance Gas (HRG) model result.

From talk by Volodymyr Vovchenko ＠QM2018

Taylor展開法と虚数化学ポテンシャル法の一致

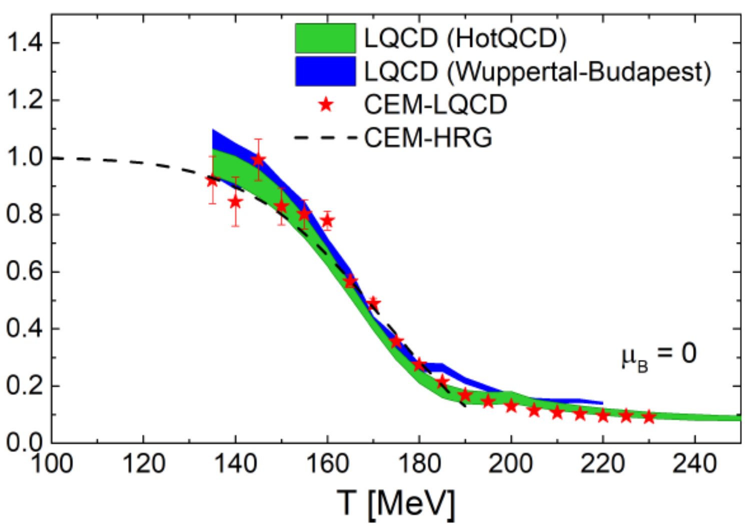
χ_2^B



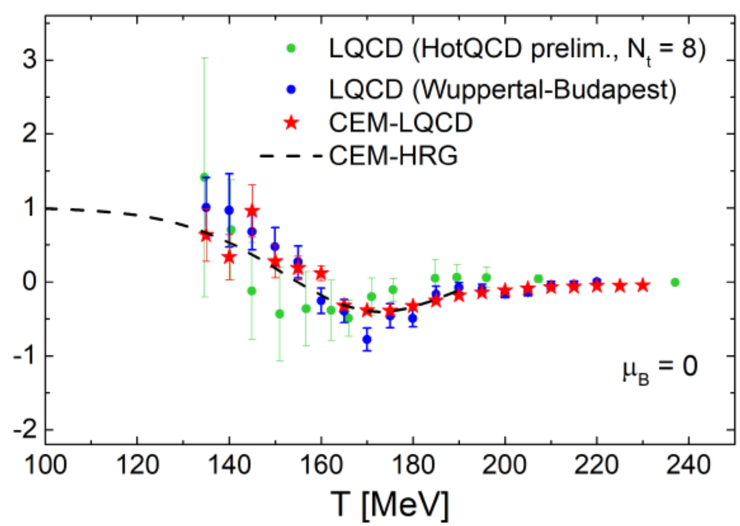
Hot QCD(green): Taylor展開法

Wuppertal-Budapest(blue): 虚数化学potential

χ_4^B / χ_2^B



χ_6^B / χ_2^B



Cluster Expansion Model (CEM)

Model formulation:

- Fugacity expansion for baryon number density

$$\frac{\rho_B(T,\mu_B)}{T^3} = \chi_1^B(T,\mu_B) = \sum_{k=1}^{\infty} b_k(T)\,\sinh(k\mu_B/T)$$

- $b_1(T)$ and $b_2(T)$ are model input
- All higher order coefficients are predicted: $b_k(T) = \alpha_k^{SB}\,\frac{[b_2(T)]^{k-1}}{[b_1(T)]^{k-2}}$

Physical picture: Hadron gas with repulsion at moderate *T*, "weakly" interacting quarks and gluons at high *T*

Resummed analytic form:

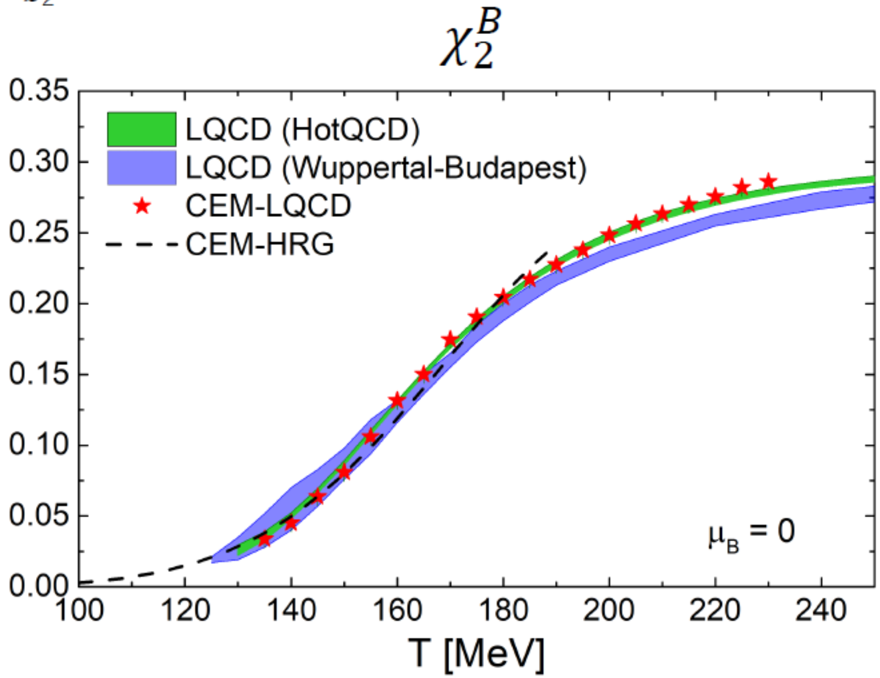
$$\frac{\rho_B(T,\mu_B)}{T^3} = -\frac{2}{27\pi^2}\frac{\hat{b}_1^2}{\hat{b}_2}\left\{4\pi^2\left[\mathrm{Li}_1(x_+) - \mathrm{Li}_1(x_-)\right] + 3\left[\mathrm{Li}_3(x_+) - \mathrm{Li}_3(x_-)\right]\right\}$$

$$\hat{b}_{1,2} = \frac{b_{1,2}(T)}{b_{1,2}^{\mathrm{SB}}}, \quad x_{\pm} = -\frac{\hat{b}_2}{\hat{b}_1}e^{\pm\mu_B/T}, \quad \mathrm{Li}_s(z) = \sum_{k=1}^{\infty}\frac{z^k}{k^s}$$

From talk by Volodymyr Vovchenko @QM2018

CEM: Baryon number susceptibility

$$\chi_k^B(T,\mu_B) = -\frac{2}{27\pi^2}\frac{\hat{b}_1^2}{\hat{b}_2}\left\{4\pi^2\left[\mathrm{Li}_{2\text{-}k}(x_+) + (-1)^k\,\mathrm{Li}_{2\text{-}k}(x_-)\right] + 3\left[\mathrm{Li}_{4\text{-}k}(x_+) + (-1)^k\,\mathrm{Li}_{4\text{-}k}(x_-)\right]\right\}$$



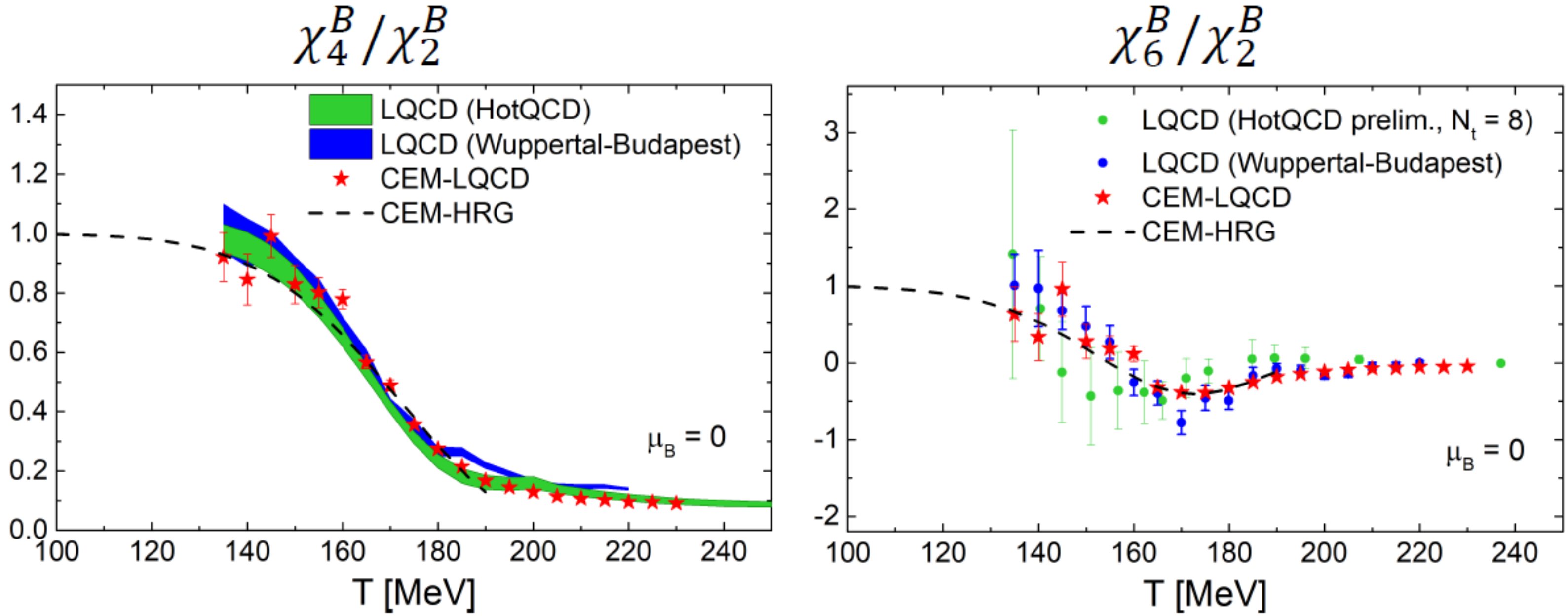
Lattice data from 1112.4416 (Wuppertal-Budapest), 1701.04325 (HotQCD)

Model inputs used:

- CEM-LQCD: $b_1(T)$ and $b_2(T)$ from LQCD simulations at imaginary μ_B
- CEM-HRG: $b_1(T)$ and $b_2(T)$ from excluded-volume HRG

CEM: Higher-order susceptibilities

$$\chi_k^B(T,\mu_B) = -\frac{2}{27\pi^2}\frac{\hat{b}_1^2}{\hat{b}_2}\left\{4\pi^2\left[\mathrm{Li}_{2\text{-}k}(x_+) + (-1)^k\,\mathrm{Li}_{2\text{-}k}(x_-)\right] + 3\left[\mathrm{Li}_{4\text{-}k}(x_+) + (-1)^k\,\mathrm{Li}_{4\text{-}k}(x_-)\right]\right\}$$



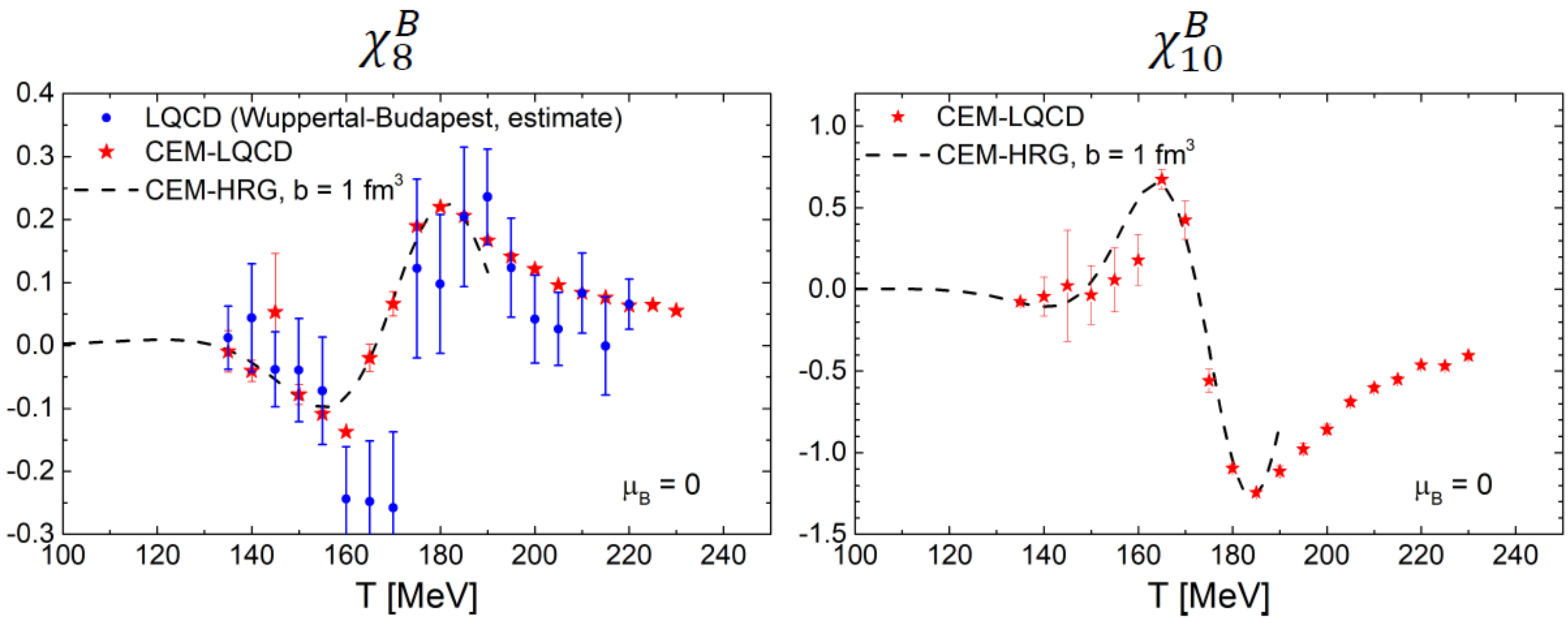
Lattice data from 1805.04445 (Wuppertal-Budapest), 1701.04325 & 1708.04897 (HotQCD)

Lattice data on higher-order susceptibilities validate CEM

From talk by Volodymyr Vovchenko @QM2018

CEM: Higher-order susceptibilities

$$\chi_k^B(T,\mu_B) = -\frac{2}{27\pi^2}\frac{\hat{b}_1^2}{\hat{b}_2}\left\{4\pi^2\left[\mathrm{Li}_{2\text{-}k}(x_+) + (-1)^k\ \mathrm{Li}_{2\text{-}k}(x_-)\right] + 3\left[\mathrm{Li}_{4\text{-}k}(x_+) + (-1)^k\ \mathrm{Li}_{4\text{-}k}(x_-)\right]\right\}$$



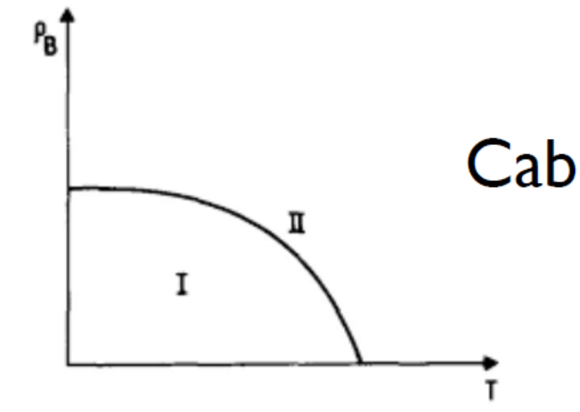
Preliminary lattice estimate from 1805.04445 (Wuppertal-Budapest)

To be verified by future lattice data

Cluster expansion model (CEM) combines hadron gas with deconfinement and is consistent with presently available lattice data, both at μ=0 and imaginary μ_B

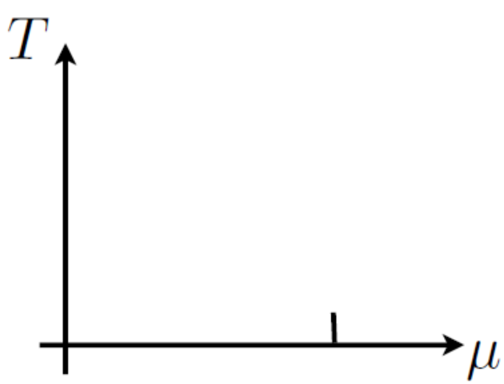
From talk by Philip De Forcrand ＠NFQCD2018

Time evolution of the phase diagram of QCD



Cabibbo & Parisi, 1975

"little bang"



2020

"no bang at all" ?

※個人の感想です

※ありうるシナリオの内の１つです

Many proposed methods to solve(avoid) the sign problem

Proposed methods	Based on
Taylor expansion method	Monte Carlo method
Canonical approach	Monte Carlo method
Reweighting method	Monte Carlo method
with imaginary chemical potential	Monte Carlo method
$\mathrm{SU}(2)_c$ QCD	Monte Carlo method
Lefschetz thimble decomposition	Cauchy's integral formula
Holomorphic gradient flow method	Cauchy's integral formula
path optimization method	Cauchy's integral formula
Complex Langevin method	Stochastic quantization

まだ理論的に完成していない所もあるが、高密度領域でも計算できる可能性のある手法

and so on

Many proposed methods to solve(avoid) the sign problem

Proposed methods	Based on
Taylor expansion method	Monte Carlo method
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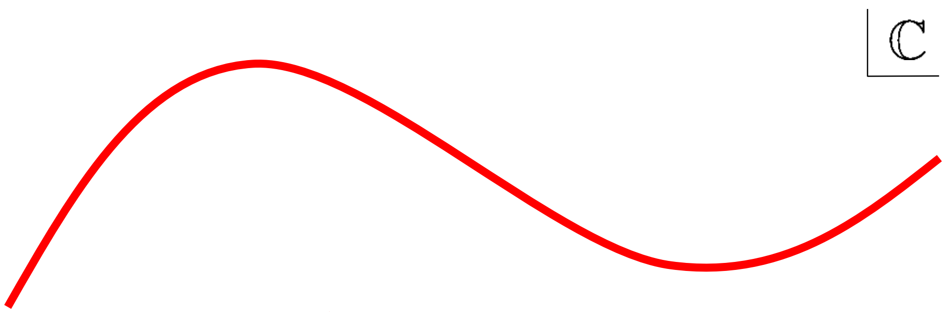
まだ理論的に完成していない所もあるが、高密度領域でも計算できる可能性のある手法

and so on

積分経路を複素化

Cauchy's integration theorem: We can deform integration path.
(because usual $e^{-\mathrm{Re}S}$ exponentially damp at infinity)



γ : Arbitrary deformed integration path

Complexify variable $x \in \mathbb{R} \to z \in \mathbb{C}$

Real axis = Original integration path

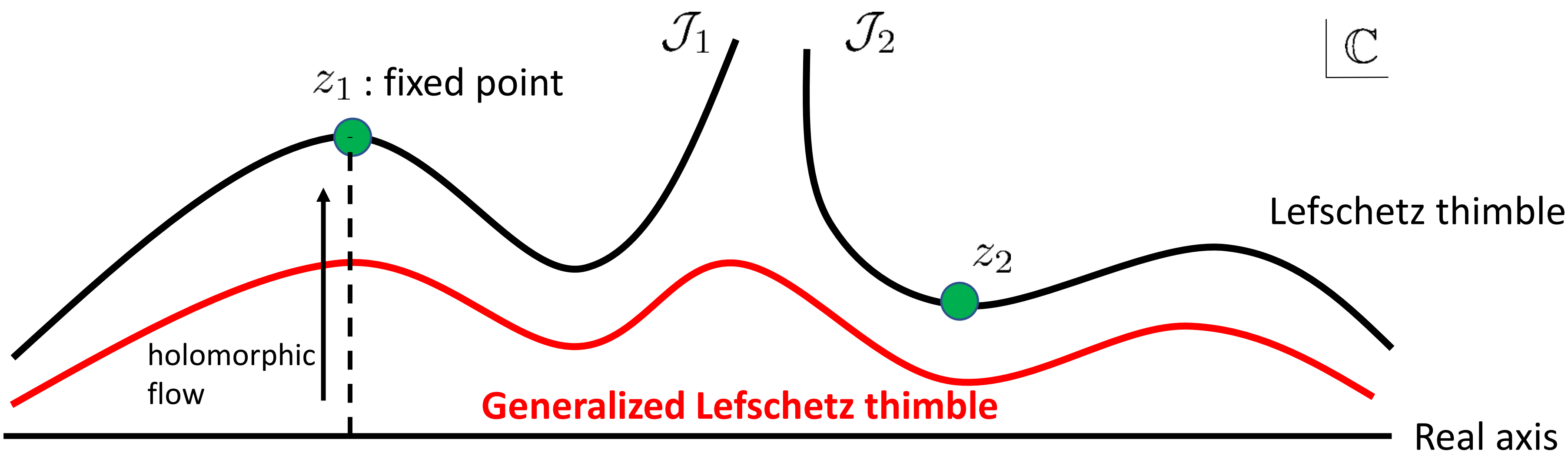
$$\int_{-\infty}^{\infty} dx e^{-S(x)} = \int_{\gamma} dz e^{-S(z)}$$

The sign problem (or oscillation) of integrand are different, but the values of the integrals are same.

Question: What is the “best” contour γ ?

“best” contour γ の候補

Integration path	sign problem itself	elgodicity	model space of γ
Original path = Real axis	×	○	
Lefschetz thimble F. Pham (1983),　E. Witten (2010)	◎	×	
Generalized Lefschetz thimble Alexandru, et al. (2016).	○	○	○
Path optimization method Yuto Mori, Kouji Kashiwa, Akira Ohnishi (2017).	○	○	◎(No Ansatz)



※実際はreweighting methodと組み合わせて使う

Many proposed methods to solve(avoid) the sign problem

Proposed methods	Based on
Taylor expansion method	Monte Carlo method
Canonical approach	
Reweighting method	
with imaginary chemical potential	
$\mathrm{SU(2)_c}$ QCD	
Lefschetz thimble decomposition	Cauchy's integral formula
Holomorphic gradient flow method	
path optimization method	
Complex Langevin method	Stochastic quantization

まだ理論的に完成していない所もあるが、高密度領域でも計算できる可能性のある手法

and so on

Complex Langevin method

G. Parisi, Phys. Lett. B131, 393 (1983).
J. R. Klauder, J. Stat. Phys. 39, 53 (1985).
J. Ambjorn and S. K. Yang, Phys. Lett. B165, 140 (1985).

Consider a theory with complex action S(x) and real variable x

$$Z = \int dx\ \mathrm{e}^{-S(x)} \quad , \qquad x \in \mathbb{R}, \quad S(x) \in \mathbb{C}$$

Replace the real variable x with complex variable z;

$$x \in \mathbb{R} \to z \in \mathbb{C}$$

Introduce a fictitious time t and suppose that the variable z satisfies the Langevin eq;

$$\frac{dz}{dt} = -\frac{\partial S(z)}{\partial z} + \eta \ , \qquad \langle \eta(t) \rangle = 0, \quad \langle \eta(t)\eta(t') \rangle = 2\delta(t-t')$$

η:real Gaussian noise
$\langle \dots \rangle$:noise average

Noise average of an observable agrees with the quantum average at infinite fictitious time**?**;

$$\langle \mathcal{O}(z(t)) \rangle \overset{?}{\to} \frac{1}{Z}\int dx\ \mathcal{O}\mathrm{e}^{-S(x)} \qquad (t \to \infty)$$

$x \in \mathbb{R}, \quad S(x) \in \mathbb{R}$ の時は正しい事が証明されている

Complex Langevin method

$$\langle \mathcal{O}(z(t)) \rangle \overset{?}{\to} \frac{1}{Z} \int dx\ \mathcal{O}(x) \mathrm{e}^{-S(x)} \qquad (t \to \infty)$$

Q. Is stochastic quantization correct even with complex action?

A. It depends. Sometimes, complex Langevin method gives the incorrect result.

Some criteria for correctness of the complex Langevin dynamics are proposed.

- Many(infinite) identities to be checked

G. Aarts, E. Seiler, and I.-O. Stamatescu, Phys.Rev. D81, 054508 (2010).
G. Aarts, F. A. James, E. Seiler, and I.-O. Stamatescu, Eur.Phys.J. C71, 1756 (2011).

Improve ↓

- Reliability test by the probability distribution of the drift term of the complex Langevin eq.

J. Nishimura and S. Shimasaki, Phys. Rev. D92, 011501 (2015).

K. Nagata, J. Nishimura, S. Shimasaki, Phys. Rev. D94, 114515 (2016).

Complex Langevin equation: $$\frac{dz}{dt} = -\frac{\partial S(z)}{\partial z} + \eta$$

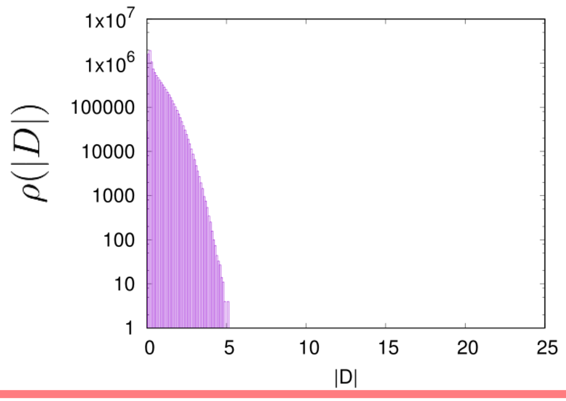
Drift term D(z)

Complex Langevin methodが正しい場合の条件

K. Nagata, J. Nishimura, S. Shimasaki, Phys. Rev. D94, 114515 (2016).

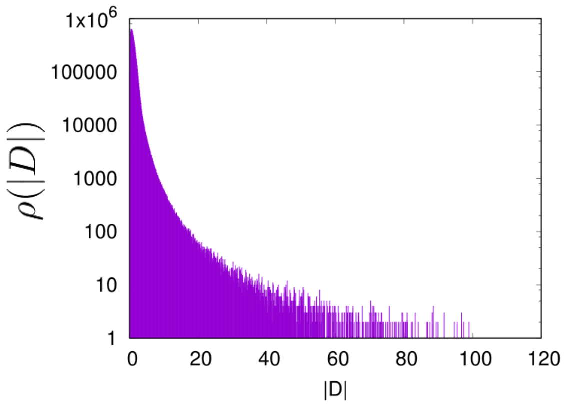
Distribution of Drift **exponentially** damps

⇒ Complex Langevin method works well.
(correct convergence)



Not exponentially damps

⇒ Complex Langevin method does not work.
(wrong convergence)



Complex Langevin equation: $$\frac{dz}{dt} = -\frac{\partial S(z)}{\partial z} + \eta$$

Drift term D(z)

Complex Langevin methodの特徴

長所　・計算コストが低い(Algorithm自体はMonte Carloとかなり似ている)
　　　・$\mu/T \geq 1$でも使える（使えない理由は特に無い）
　　（・正しいかどうかはさておき計算は収束する）

短所　・限界がある（ μ/T で決まるとは限らない）
　　　・一般に相転移近傍で計算がうまくいかなくなるが、
　　　　そうじゃない時もあってよくわからない
　　　・complex Langevin methodの正当性は厳密な意味では証明されていない
　　　　（物理学者的にはOKかもしれない）

計算コストが低い事から、QCDにも適用されている
- Swansea university group
- **KEK group (Yuta Ito, Hideo Matsufuru, Kanto Moritake, Jun Nishimura, Shinji Shimasaki, Asato Tsuchiya, Shoichiro Tsutsui)**

Setup: low temperature region

- N_f = 4, staggered fermion
- Lattice size: $8^3 \times 16$

cf) $4^3 \times 8$ results in J. Nishimura's talk

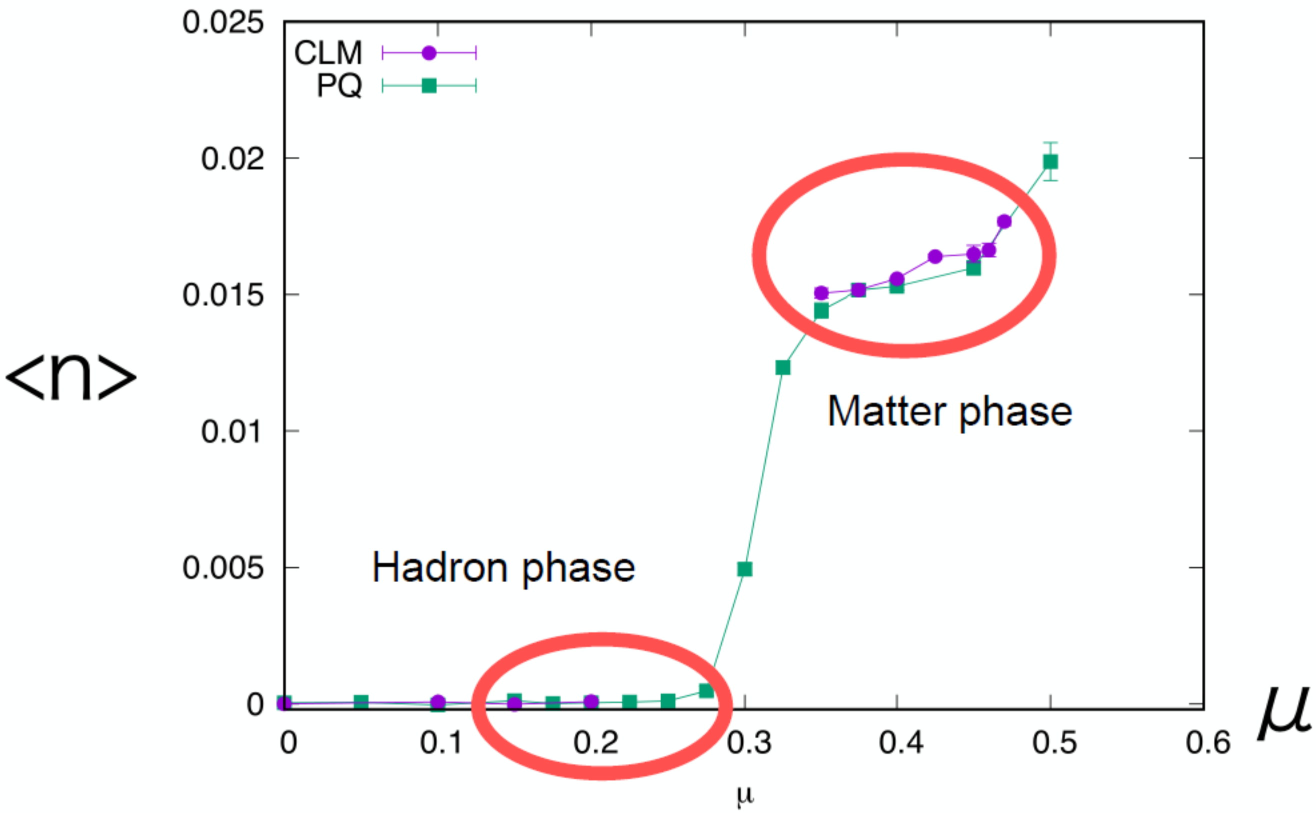
- β = 5.7
- μa= 0.0 – 0.5
- Quark mass: $m_q a$ = 0.01, 0.05
- Langevin steps = 10^5 – 10^6
- Computational resource: K computer

We compare the results with the RHMC results of the phase quenched (PQ) simulation.

From talk by Shoichiro Tsutsui @NFQCD2018

Comparison with PQ simulation

m = 0.05

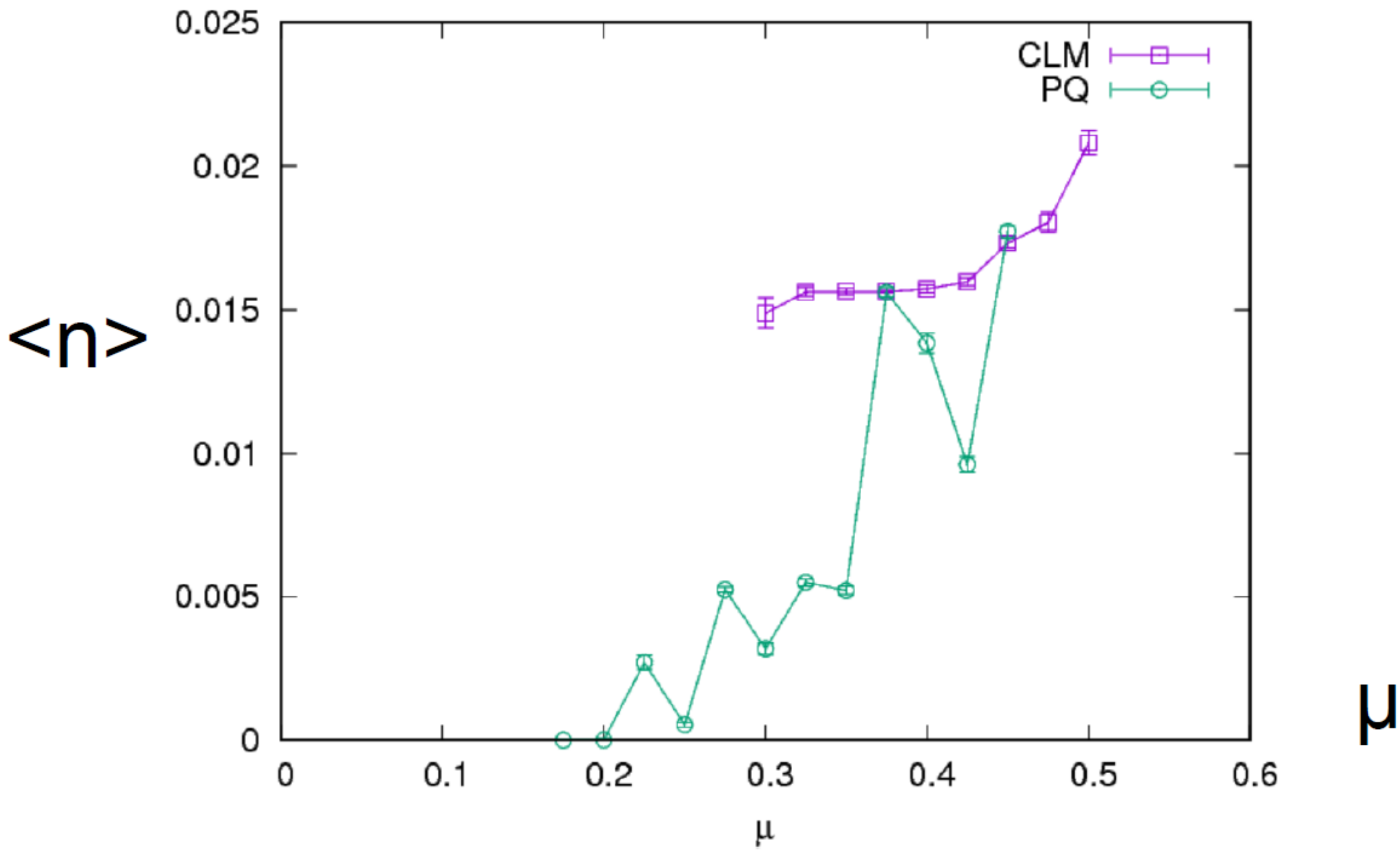


Qualitative difference is not observed → due to too heavy pion?

From talk by Shoichiro Tsutsui @NFQCD2018

Comparison with PQ simulation

m = 0.01

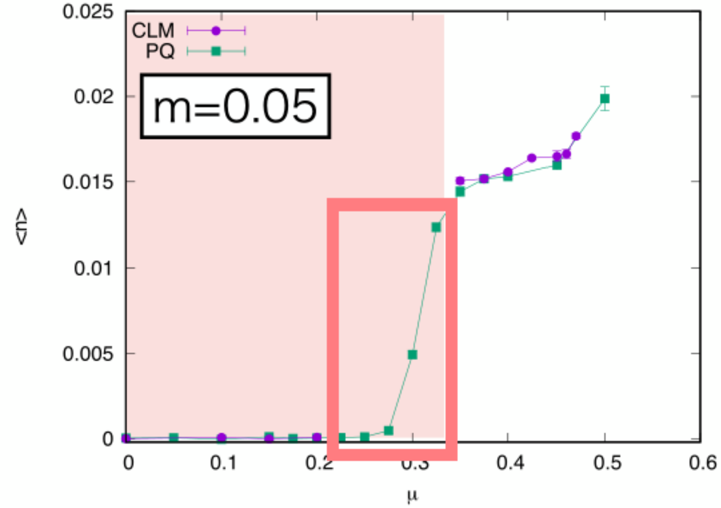


From talk by Shoichiro Tsutsui @NFQCD2018

m = 0.01 vs 0.05 (μ<0.3)

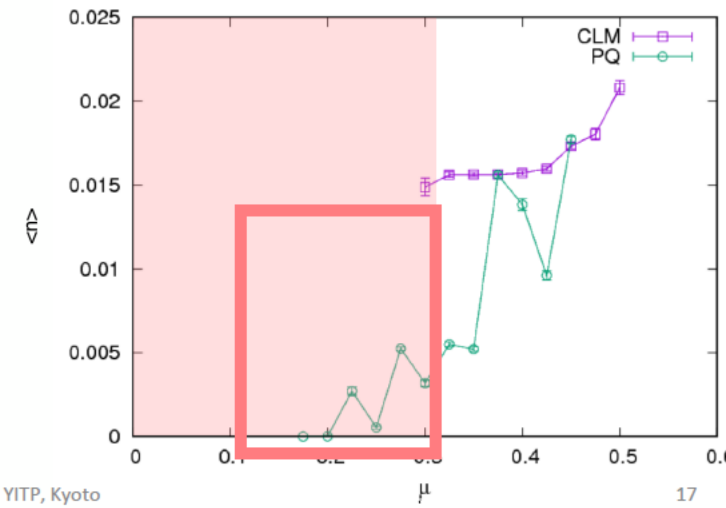
PQ

As m=0.05 → 0.01, critical chemical potential lowers.



CL

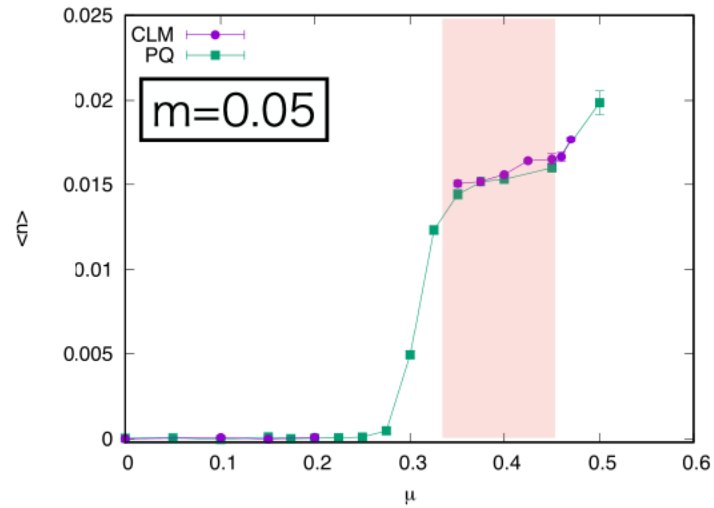
Region where the singular drift problem occurs depends on mass.



m = 0.01 vs 0.05 (μ>0.3)

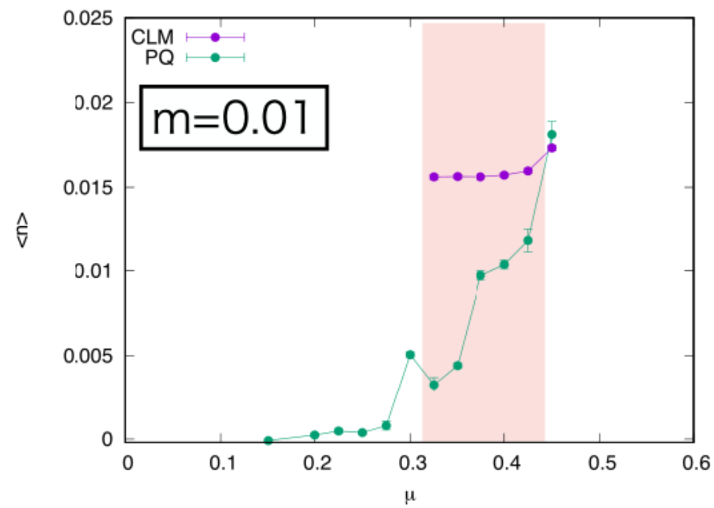
PQ

<n> (~ pion density) is sensitive to the change of mass. (finite size effect may arise)



CL

<n> (= baryon number density) is not sensitive.

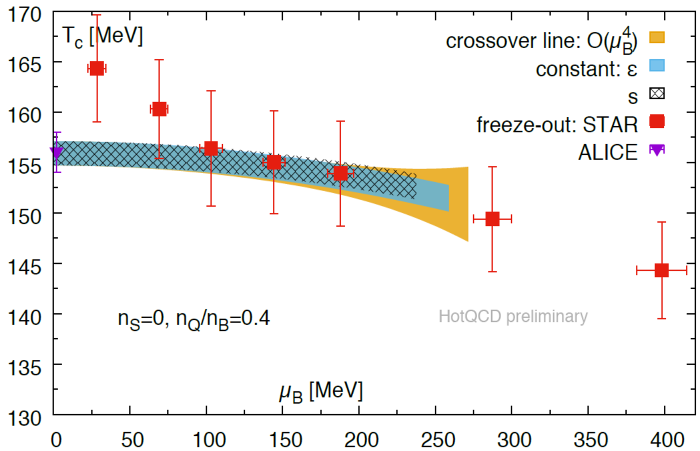


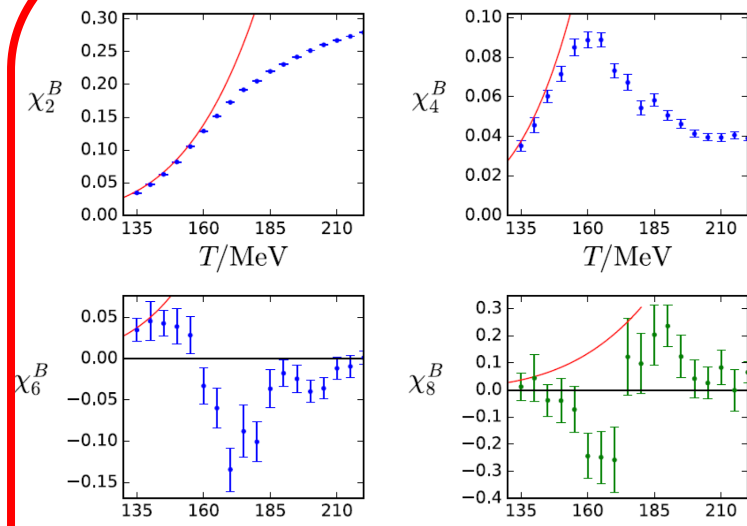
CLM succeeded to take into account the complex phase of the fermion determinant.

Summary①

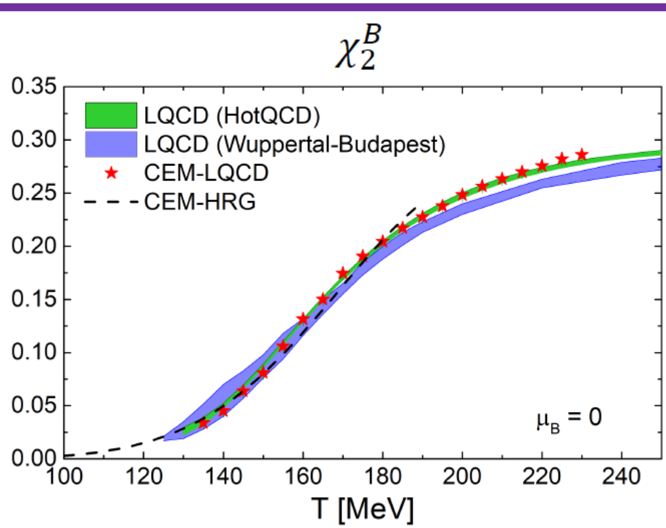
現状：QCDの符号問題は難しいが、
①とりあえず確実な方法を試してみた

by Taylor展開法





by 虚数化学ポテンシャル



- Taylor展開法と虚数化学ポテンシャルがconsistent
- Taylor展開法と虚数化学ポテンシャル、どちらでもcritical pointの兆候は見えず
- latticeの結果をよくFitできるmodelが提案されている（何を意味するかは非自明）

Summary②

現状：QCDの符号問題は難しいが、
②頑張って解こうとしている人もいる。
QCDの結果が出つつあるが、実験と比較できるようなものはまだ難しい。
何かbreak throughが起こればいけるかも？~~（3,40年前から言われている）~~

Proposed methods	Based on	Disadvantage
Taylor expansion method	Monte Carlo method	展開に頼る
Canonical approach		展開に頼る
Reweighting method		数値計算コスト大
with imaginary chemical potential		展開に頼る
$\mathrm{SU}(2)_{\mathrm{c}}$ **QCD**		そもそもQCDじゃない
Lefschetz thimble decomposition	Cauchy's integral formula	elgodicity, global sign problem
Holomorphic gradient flow method		計算コスト大、global sign problem
path optimization method		計算コスト大？、global sign problem
Complex Langevin method	Stochastic quantization	基本的に相転移が苦手

これから：これ以上Taylor展開の高次の係数は求められないので、
展開に頼らないかつ**効率的な**方法を考えなければならない。頑張ります...